

# On standard derived equivalences of orbit categories

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## Abstract

Let  $\mathbf{k}$  be a commutative ring,  $\mathcal{A}$  and  $\mathcal{B}$  – two  $\mathbf{k}$ -linear categories with an action of a group  $G$ . We introduce the notion of a standard  $G$ -equivalence from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$ . We construct a map from the set of standard  $G$ -equivalences to the set of standard equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$  and a map from the set of standard  $G$ -equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$  to the set of standard equivalences from  $\mathcal{K}_p^b(\mathcal{B}/G)$  to  $\mathcal{K}_p^b(\mathcal{A}/G)$ . We investigate the properties of these maps and apply our results to the case where  $\mathcal{A} = \mathcal{B} = R$  is a Frobenius  $\mathbf{k}$ -algebra and  $G$  is the cyclic group generated by its Nakayama automorphism  $\nu$ . We apply this technique to obtain the generating set of the derived Picard group of a Frobenius Nakayama algebra over an algebraically closed field.

## 1 Introduction

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two derived equivalent categories. The notion of a standard equivalence from  $\mathcal{DB}$  to  $\mathcal{DA}$  was introduced in [1]. This notion generalizes the notion of a standard equivalence for algebras [3]. We define such standard equivalences in terms of tilting subcategories instead of tilting complexes of bimodules. We denote by  $\text{TrPic}(\mathcal{A}, \mathcal{B})$  the set of standard equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$ , standard equivalences from  $\mathcal{DB}$  to  $\mathcal{DA}$  correspond bijectively to standard equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$ . In [1] it is proved that the composition of standard equivalences and the inverse equivalence of a standard equivalence are again standard. In particular, composition defines a group structure on  $\text{TrPic}(\mathcal{A}) = \text{TrPic}(\mathcal{A}, \mathcal{A})$ . We call this group the derived Picard group of  $\mathcal{A}$ . In the case where a group  $G$  acts on  $\mathcal{A}$  and  $\mathcal{B}$ , we introduce the notion of a standard  $G$ -equivalence from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$ . We denote by  $\text{TrPic}_G(\mathcal{A}, \mathcal{B})$  the set of such equivalences. It appears that the composition of standard  $G$ -equivalences is defined and it determines a group structure on  $\text{TrPic}_G(\mathcal{A}) = \text{TrPic}_G(\mathcal{A}, \mathcal{A})$ . We construct the maps  $\Phi_{\mathcal{A}, \mathcal{B}} : \text{TrPic}_G(\mathcal{A}, \mathcal{B}) \rightarrow \text{TrPic}(\mathcal{A}, \mathcal{B})$  and  $\Psi_{\mathcal{A}, \mathcal{B}} : \text{TrPic}_G(\mathcal{A}, \mathcal{B}) \rightarrow \text{TrPic}(\mathcal{A}/G, \mathcal{B}/G)$  which respect the composition. Here  $\mathcal{A}/G$  is the orbit category defined in [2]. We investigate the properties of these maps. We prove that  $\Phi_{\mathcal{A}, \mathcal{B}}$  sends a standard  $G$ -equivalence to a Morita equivalence iff  $\Psi_{\mathcal{A}, \mathcal{B}}$  sends this standard  $G$ -equivalence to a Morita equivalence. We prove a theorem which gives a necessary and sufficient condition for an element of  $\text{TrPic}(\mathcal{A}, \mathcal{B})$  to

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lie in the image of  $\Phi_{\mathcal{A},\mathcal{B}}$ . In the case of a finite group  $G$  we provide necessary and sufficient condition for an element of  $\text{TrPic}(\mathcal{A}/G, \mathcal{B}/G)$  to lie in the image of  $\Psi_{\mathcal{A},\mathcal{B}}$ .

It was proved in [3] that the Nakayama functor commutes with any standard derived equivalence. Suppose that  $R$  is a Frobenius algebra with a Nakayama automorphism  $\nu$ . Suppose that  $\text{ord } \nu = n < \infty$ . Then the cyclic group  $G = \langle \nu \rangle \cong C_n$  acts on  $R$  and we can define an algebra  $R/G$ . We prove that the homomorphism  $\Phi_R = \Phi_{R,R} : \text{TrPic}_G(R) \rightarrow \text{TrPic}(R)$  is an epimorphism if for any  $a \in Z(R)^*$  there is an element  $b \in Z(R)^*$  such that  $a = b^n$ . Moreover, we prove that  $\text{Cok}(\Phi_R)$  is generated by the classes of elements from the Picard group of  $R$  if for any  $a \in Z(R)^*$  there is an element  $b \in R^*$  and an automorphism  $\sigma$  of  $R$  such that  $a = b\nu(b) \dots \nu^{n-1}(b)$  and  $\sigma\nu\sigma^{-1}(x) = \nu(bxb^{-1})$  for all  $x \in R$ . We apply these facts to find a generating set of the derived Picard group of a Frobenius Nakayama algebra using the generating set of the derived Picard group of a symmetric Nakayama algebra from [4].

## 2 Standard equivalences as tensor products

Throughout this paper  $\mathbf{k}$  is a fixed commutative ring. We assume everywhere that  $\mathcal{A}$  and  $\mathcal{B}$  are small,  $\mathbf{k}$ -linear and  $\mathbf{k}$ -flat categories ( $\mathcal{A}$  is called  $\mathbf{k}$ -flat if  $\mathcal{A}(x, y)$  is a flat  $\mathbf{k}$ -module for any  $x, y \in \mathcal{A}$ ). We simply write  $\otimes$  for  $\otimes_{\mathbf{k}}$ . In this section we recall some basic definitions and results on standard equivalences of derived categories.

**Definition 1.** Contravariant functors from  $\mathcal{A}$  to  $\text{Mod } \mathbf{k}$  are called  $\mathcal{A}$ -modules. We denote by  $\text{Mod } \mathcal{A}$  the category of  $\mathcal{A}$ -modules. An  $\mathcal{A}$ -module is called *projective* if it is a direct summand of a direct sum of representable functors (a *representable functor* is a functor isomorphic to  $\mathcal{A}(-, x)$  for some  $x \in \mathcal{A}$ ). An  $\mathcal{A}$ -module is called *finitely generated* if it is an epimorphic image of a finite direct sum of representable functors.

A morphism of  $\mathcal{A}$ -modules  $d : V \rightarrow V$  is called differential if  $d^2 = 0$ . A  $\mathbb{Z}$ -graded  $\mathcal{A}$ -module is a module  $V$  with a decomposition  $V = \bigoplus_{n \in \mathbb{Z}} V_n$ . We say that a morphism  $f : V \rightarrow V'$  is of degree  $m$  if  $f = \sum_{n \in \mathbb{Z}} f_n$  for some morphisms  $f_n : V_n \rightarrow V'_{n+m}$ . We denote by  $V[m]$  the module  $V$  with the following grading:  $V[m]_n = V_{n+m}$ .

**Definition 2.** An  $\mathcal{A}$ -complex is a  $\mathbb{Z}$ -graded module  $V$  with a differential  $d_V : V \rightarrow V$  of degree 1. A morphism of  $\mathcal{A}$ -complexes is a morphism of  $\mathcal{A}$ -modules of degree 0 which commutes with differentials. We denote by  $\mathcal{CA}$  the category of  $\mathcal{A}$ -complexes. A morphism  $f : V \rightarrow V'$  is called *null homotopic* if  $f = hd_V + d_{V'}h$  for some morphism of modules  $h : V \rightarrow V'$  of degree  $-1$ . We denote by  $H(V, V')$  the space of all null homotopic morphisms from  $V$  to  $V'$ . The *homotopy category* of  $\text{Mod } \mathcal{A}$  is the category  $\mathcal{KA}$  with the same objects as  $\mathcal{CA}$  and morphism spaces  $\mathcal{KA}(V, V') = \mathcal{CA}(V, V')/H(V, V')$ . The *derived category* of  $\text{Mod } \mathcal{A}$  denoted by  $\mathcal{DA}$  is the localization of  $\mathcal{KA}$  at the set of quasi-isomorphisms (a morphism from  $V$  to  $V'$  is called a quasi-isomorphism if it induces an isomorphism from  $\text{Ker } d_V/\text{Im } d_V$  to  $\text{Ker } d_{V'}/\text{Im } d_{V'}$ ).

We denote by  $\mathcal{K}_p\mathcal{A}$  the full subcategory of  $\mathcal{KA}$  consisting of all projective complexes. The canonical functor from  $\mathcal{KA}$  to  $\mathcal{DA}$  induces an equivalence between categories  $\mathcal{K}_p\mathcal{A}$  and  $\mathcal{DA}$ . We denote by  $\mathcal{K}_p^b\mathcal{A}$  the full subcategory of  $\mathcal{KA}$  consisting of all finitely generated projective complexes  $V$  such that  $V_n = 0$  for large enough and small enough  $n \in \mathbb{Z}$ .

**Definition 3.** A full subcategory  $\mathcal{X}$  of  $\mathcal{K}_p^b \mathcal{A}$  is called a *tilting subcategory* for  $\mathcal{A}$  if

- $\mathcal{K}_p^b \mathcal{A}(U, V[i]) = 0$  for all  $U, V \in \mathcal{X}$ ,  $i \neq 0$ ,
- $\mathcal{A}(-, x)$  ( $x \in \mathcal{A}$ ) lies in the smallest full triangulated subcategory of  $\mathcal{K}_p^b \mathcal{A}$  containing  $\mathcal{X}$  and closed under isomorphisms and direct summands (we denote this subcategory of  $\mathcal{K}_p^b \mathcal{A}$  by  $\text{thick} \mathcal{X}$ ).

**Definition 4.** The *tensor product*  $\mathcal{A} \otimes \mathcal{B}$  of  $\mathcal{A}$  and  $\mathcal{B}$  is a  $\mathbf{k}$ -linear category defined in the following way. Its objects are pairs  $(x, y)$  where  $x \in \mathcal{A}$ ,  $y \in \mathcal{B}$ . Its morphism spaces are

$$(\mathcal{A} \otimes \mathcal{B})((x_1, y_1), (x_2, y_2)) = \mathcal{A}(x_1, x_2) \otimes \mathcal{B}(y_1, y_2).$$

The composition in  $\mathcal{A} \otimes \mathcal{B}$  is given by the formula

$$(f_2 \otimes g_2)(f_1 \otimes g_1) = f_2 f_1 \otimes g_2 g_1,$$

where  $f_1 \in \mathcal{A}(x_1, x_2)$ ,  $f_2 \in \mathcal{A}(x_2, x_3)$ ,  $g_1 \in \mathcal{B}(y_1, y_2)$ ,  $g_2 \in \mathcal{B}(y_2, y_3)$  and  $x_1, x_2, x_3 \in \mathcal{A}$ ,  $y_1, y_2, y_3 \in \mathcal{B}$ .

Let  $X$  be an  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex. It defines a functor  $T_X : \mathcal{CB} \rightarrow \mathcal{CA}$  in the following way. If  $M \in \mathcal{CB}$ , then  $(T_X M)(x)$  ( $x \in \mathcal{A}$ ) is the cokernel of the map

$$\rho_{X,M}(x) : \bigoplus_{y,z \in \mathcal{B}} (M(z) \otimes \mathcal{B}(y, z) \otimes X(x, y)) \rightarrow \bigoplus_{y \in \mathcal{B}} (M(y) \otimes X(x, y))$$

defined by the equality

$$\rho_{X,M}(x)(u \otimes g \otimes v) = M(g)(u) \otimes v - u \otimes X(\text{Id}_x \otimes g)(v)$$

for  $u \in M(z)$ ,  $v \in X(x, y)$ ,  $g \in \mathcal{B}(y, z)$ . If  $f \in \mathcal{A}(x_1, x_2)$  ( $x_1, x_2 \in \mathcal{A}$ ), then  $(T_X M)(f)$  is the unique map such that the diagram

$$\begin{array}{ccc} \bigoplus_{y \in \mathcal{B}} (M(y) \otimes X(x_2, y)) & \longrightarrow & (T_X M)(x_2) \\ \bigoplus_{y \in \mathcal{B}} (\text{Id}_{M(y)} \otimes X(f \otimes \text{Id}_y)) \downarrow & & \downarrow (T_X M)(f) \\ \bigoplus_{y \in \mathcal{B}} (M(y) \otimes X(x_1, y)) & \longrightarrow & (T_X M)(x_1) \end{array}$$

commutes. Finally, for  $\alpha \in \mathcal{CB}(M, N)$  we obtain  $T_X \alpha$  from the commutativity of diagrams

$$\begin{array}{ccc} \bigoplus_{y \in \mathcal{B}} (M(y) \otimes X(x, y)) & \longrightarrow & (T_X M)(x) \\ \bigoplus_{y \in \mathcal{B}} (\alpha_y \otimes \text{Id}_{X(x, y)}) \downarrow & & \downarrow (T_X \alpha)_x \\ \bigoplus_{y \in \mathcal{B}} (N(y) \otimes X(x, y)) & \longrightarrow & (T_X N)(x) \end{array}$$

We denote by  $\mathbf{L}T_X$  the left derived functor of  $T_X$ , i.e. the composition

$$\mathcal{DB} \xrightarrow{\mathbf{p}} \mathcal{K}_p \mathcal{B} \xrightarrow{T_X} \mathcal{KA} \xrightarrow{\pi} \mathcal{DA},$$

where  $\mathbf{p}$  sends an object of  $\mathcal{DB}$  to its projective resolution (see [1, Theorem 3.1] for details) and  $\pi$  is the canonical functor from  $\mathcal{KA}$  to  $\mathcal{DA}$ . If  $X$  is an  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex, then for any  $y \in \mathcal{B}$  we can define an  $\mathcal{A}$ -complex  $X^y$  as follows:  $X^y(x) = X(x, y)$  and  $X^y(f) = X(f \otimes \text{Id}_y)$  for objects and morphisms of  $\mathcal{A}$  respectively. Then  $\mathbf{L}T_X$  is an equivalence iff the following conditions hold:

- $X^y$  is isomorphic to some object of  $\mathcal{K}_p^b \mathcal{A}$  in  $\mathcal{KA}$  for any  $y \in \mathcal{B}$ ,
- the full subcategory of  $\mathcal{K}_p^b \mathcal{A}$  consisting of objects isomorphic to some  $X^y$  ( $y \in \mathcal{B}$ ) in  $\mathcal{KA}$  is a tilting subcategory for  $\mathcal{A}$ ,
- the map  $\mathcal{B}(y, z) \rightarrow \mathcal{KA}(X^y, X^z)$  is an isomorphism for all  $y, z \in \mathcal{B}$ .

Moreover,  $\mathbf{LT}_X \cong \mathbf{LT}_Y$  iff  $X \cong Y$  in  $\mathcal{D}(\mathcal{A} \otimes \mathcal{B}^{\text{op}})$ . See [1, 6.1] for details. If  $\mathbf{LT}_X$  is an equivalence, then  $X$  is called a *tilting  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex*.

If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a functor, then we denote by  $F(\mathcal{A})$  the full subcategory of  $\mathcal{B}$  formed by objects isomorphic to some  $F(U)$  ( $U \in \mathcal{A}$ ).

**Definition 5.** We call  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  a *tilting functor* if  $\theta(\mathcal{B})$  is a tilting subcategory for  $\mathcal{A}$  and  $\theta$  induces an equivalence from  $\mathcal{B}$  to  $\theta(\mathcal{B})$ .

Categories  $\mathcal{A}$  and  $\mathcal{B}$  are said to be *derived equivalent* if the derived categories  $\mathcal{DA}$  and  $\mathcal{DB}$  are equivalent as triangulated categories. The following theorem is well known (see [1, 9.2, Corollary] and [2, Theorem 4.6]).

**Theorem 1.** *The following conditions are equivalent*

- (1)  $\mathcal{A}$  and  $\mathcal{B}$  are derived equivalent;
- (2) there is a tilting  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex  $X$ ;
- (3) there is a tilting functor  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$ .

**Remark 1.** Note that if  $X$  is a tilting  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex, then the objects of  $\mathcal{K}_p^b \mathcal{A}$  isomorphic to  $X^y$  ( $y \in \mathcal{B}$ ) form a tilting subcategory which is equivalent to  $\mathcal{B}$ . We denote the corresponding tilting functor (which is defined modulo natural isomorphism) by  $\theta_X$ . Conversely, if we have a tilting functor  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$ , then we can construct a tilting  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex  $X$  such that  $\theta_X \cong \theta$  (see [1, Section 9]).

**Definition 6.** We call an equivalence  $F : \mathcal{DB} \rightarrow \mathcal{DA}$  *standard* if there is some  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex  $X$  such that  $F \cong \mathbf{LT}_X$ . We denote by  $\text{TrPic}(\mathcal{A}, \mathcal{B})$  the set of all standard equivalences from  $\mathcal{DB}$  to  $\mathcal{DA}$  modulo natural isomorphisms.

Let  $\mathbf{LT}_X$  be a standard equivalence. Define a  $\mathcal{B} \otimes \mathcal{A}^{\text{op}}$ -complex  $X^T$  as follows:

$$X^T(y, x) = \text{Mod}\mathcal{A}(X^y, \mathcal{A}(-, x)), \quad X^T(g \otimes f) = \text{Mod}\mathcal{A}(X(\text{Id}_- \otimes g), \mathcal{A}(-, f))$$

for  $x \in \mathcal{A}$ ,  $y \in \mathcal{B}$ , a morphism  $f$  in  $\mathcal{A}$  and a morphism  $g$  in  $\mathcal{B}$ . By [1, 6.2, Lemma] the functor  $\mathbf{LT}_{X^T}$  is quasi-inverse to  $\mathbf{LT}_X$ . Moreover, if  $\mathbf{LT}_X$  and  $\mathbf{LT}_Y$  are standard equivalences (which can be composed), then by [1, 6.3, Lemma] we have  $\mathbf{LT}_X \mathbf{LT}_Y \cong \mathbf{LT}_Z$ , where  $Z = T_{\mathbf{p}X} Y$  and  $\mathbf{p}X$  is the projective resolution of  $X$  over  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ .

**Definition 7.** The *derived Picard group* of  $\mathcal{A}$  is the set

$$\text{TrPic}(\mathcal{A}) := \text{TrPic}(\mathcal{A}, \mathcal{A})$$

with the operation of composition. It follows from the arguments above that it is actually a group.

Note that the functor  $\mathbf{LT}_X : \mathcal{DB} \rightarrow \mathcal{DA}$  is an equivalence iff its restriction to  $\mathcal{K}_p^b \mathcal{B}$  induces an equivalence to  $\mathcal{K}_p^b \mathcal{A}$ . Moreover,  $\mathbf{LT}_X \cong \mathbf{LT}_Y$  iff the corresponding equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$  are isomorphic. So we denote by  $\mathbf{LT}_X$  the corresponding equivalence from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$  too. From here on we consider only standard equivalences and identify the set  $\text{TrPic}(\mathcal{A}, \mathcal{B})$  with the set of standard equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$  modulo natural isomorphisms.

### 3 Standard equivalences and tilting subcategories

For a subcategory  $E$  of  $\mathcal{K}_p^b \mathcal{A}$  we denote by  $\text{add} E$  the full subcategory of  $\mathcal{K}_p^b \mathcal{A}$  consisting of direct summands of finite direct sums of copies of objects of  $E$ . Let us define a category  $\mathcal{C}^b \text{add} E$ . Objects of  $\mathcal{C}^b \text{add} E$  are objects of  $\text{add} E$  with a decomposition  $V = \bigoplus_{n \in \mathbb{Z}} V_n$  and a differential  $d_V = \sum_{n \in \mathbb{Z}} d_{V,n}$  where  $V_n \in \text{add} E$ ,  $d_{V,n} \in \mathcal{K} \mathcal{A}(V_n, V_{n+1})$  and  $V_n = 0$  for large enough and small enough  $n$ . If  $V, V' \in \mathcal{C}^b \text{add} E$ , then the set  $\mathcal{C}^b \text{add} E(V, V')$  is formed by maps  $f = \sum_{n \in \mathbb{Z}} f_n$  such that  $f_n \in \mathcal{K} \mathcal{A}(V_n, V'_n)$  and  $f d_V - d_{V'} f$  equals 0 in  $\mathcal{K} \mathcal{A}$ . A morphism  $f \in \mathcal{C}^b \text{add} E(V, V')$  is called null homotopic if  $f = h d_V + d_{V'} h$  for some  $h = \sum_{n \in \mathbb{Z}} h_n$  where  $h_n \in \mathcal{K} \mathcal{A}(V_n, V'_{n-1})$ . We denote the set of null homotopic morphisms from  $V$  to  $V'$  by  $B(V, V')$  again. Then  $\mathcal{K}^b \text{add} E$  is a category whose objects are the same as the objects of  $\mathcal{C}^b \text{add} E$  and whose morphism spaces are  $\mathcal{K}^b \text{add} E(V, V') = \mathcal{C}^b \text{add} E(V, V') / B(V, V')$ .

We denote by  $\mathcal{Y}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{K}_p^b \mathcal{A}$  the Yoneda embedding, i.e.  $\mathcal{Y}_{\mathcal{A}}(x) = \mathcal{A}(-, x)$  and  $\mathcal{Y}_{\mathcal{A}}(f) = \mathcal{A}(-, f)$  for an object  $x$  and a morphism  $f$  in  $\mathcal{A}$ . Let  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a tilting functor. Our aim is to define an equivalence  $F_{\theta} : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  in such a way that  $F_{\theta} \mathcal{Y}_{\mathcal{B}} = \theta$ . Denote by  $\mathcal{X}$  the category  $\theta(\mathcal{B})$ . Let  $\mathcal{P}_{\mathcal{B}}$  be the category of finitely generated projective  $\mathcal{B}$ -modules. Let us define an equivalence  $S : \mathcal{P}_{\mathcal{B}} \rightarrow \text{add} \mathcal{X}$ . Define  $S$  on direct sums of representable functors as follows:

$$S\left(\bigoplus_{i=1}^m \mathcal{B}(-, x_i)\right) = \bigoplus_{i=1}^m \theta(x_i).$$

If

$$f = (\mathcal{B}(-, f_{i,j}))_{1 \leq i \leq m, 1 \leq j \leq l} : \bigoplus_{i=1}^m \mathcal{B}(-, x_i) \rightarrow \bigoplus_{j=1}^l \mathcal{B}(-, y_j)$$

is a morphism in  $\mathcal{P}_{\mathcal{B}}$ , then

$$S(f) = (\theta(f_{i,j}))_{1 \leq i \leq m, 1 \leq j \leq l} : \bigoplus_{i=1}^m \theta(x_i) \rightarrow \bigoplus_{j=1}^l \theta(y_j).$$

Let us consider an arbitrary object  $U \in \mathcal{P}_{\mathcal{B}}$ . There is some direct sum of representable functors  $W_U$  such that  $U$  is a direct summand of  $W_U$ . Let  $\iota_U : U \rightarrow W_U$  and  $\pi_U : W_U \rightarrow U$  be the corresponding direct inclusion and projection (for convenience we assume that  $W_U = U$  and  $\iota_U = \pi_U = \text{Id}_U$  if  $U$  is a direct sum of representable functors). It follows from [5] that idempotents split in  $\mathcal{K}_p^b \mathcal{A}$ . In particular, they split in  $\text{add} \mathcal{X}$ . Since  $S(\iota_U \pi_U) : S(W_U) \rightarrow S(W_U)$  is an idempotent in  $\text{add} \mathcal{X}$ , there is some object  $X_U \in \text{add} \mathcal{X}$  and morphisms  $\iota'_U : X_U \rightarrow S(W_U)$  and  $\pi'_U : S(W_U) \rightarrow X_U$  such that  $\pi'_U \iota'_U = \text{Id}_{X_U}$  and  $\iota'_U \pi'_U = S(\iota_U \pi_U)$ . We define  $S(U) = X_U$ . If  $f : U \rightarrow V$  is a morphism in  $\mathcal{P}_{\mathcal{B}}$ , then we define  $S(f)$  by the formula

$$S(f) = \pi'_V S(\iota_V f \pi_U) \iota'_U.$$

It is clear that  $S$  is an equivalence. Then  $S$  induces an equivalence  $\bar{S} : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}^b \text{add} \mathcal{X}$ . Note also that  $S(\iota_U) = \iota'_U$  and  $S(\pi_U) = \pi'_U$ .

Let us now translate some results of [6] from the case of algebras to the case of categories. Since the arguments for the case of categories are analogous to the case of algebras, we omit most of the proofs and give only references to the corresponding results of [6].

Let  $V$  be an object of  $\mathcal{C}^b \text{add} \mathcal{X}$ . Then we can consider  $V$  as a bigraded module  $V = \bigoplus_{i,j \in \mathbb{Z}} V_{i,j}$  (where  $V_{i,j} = (V_j)_i$ ) with a differential  $\tau_0 = \sum_{j \in \mathbb{Z}} d_{V,j} : V \rightarrow V$  of degree  $(1, 0)$  and a morphism  $\tau_1 : V \rightarrow V$  of degree  $(0, 1)$  such that  $\tau_1|_{V_{i,j}} = (-1)^{i+j} d_V|_{V_{i,j}}$ . Such objects satisfy the conditions

- $V_{i,j} = 0$  if  $i$  or  $j$  is large enough or small enough;
- $\tau_0^2 = 0$ ;
- $\tau_0\tau_1 + \tau_1\tau_0 = 0$ ;
- $\tau_1^2 = 0$  in  $\mathcal{KA}$  if we consider  $V$  as an object of  $\text{add}\mathcal{X}$ .

We write  $(V, \tau_0, \tau_1)$  for such object. Note that for two such objects  $\mathcal{KA}(V, V'[i]) = 0$ ,  $i \neq 0$  if we consider them as objects of  $\text{add}\mathcal{X}$ . A morphism from  $(V, \tau_0, \tau_1)$  to  $(V', \tau'_0, \tau'_1)$  in  $\mathcal{C}^b\text{add}\mathcal{X}$  is a morphism  $\alpha : V \rightarrow V'$  of degree  $(0, 0)$  such that  $\alpha\tau_0 = \tau'_0\alpha$  and  $\alpha\tau_1 - \tau'_1\alpha$  is null homotopic if we consider  $V$  and  $V'$  as objects of  $\text{add}\mathcal{X}$ . Moreover, a morphism  $\alpha : (V, \tau_0, \tau_1) \rightarrow (V', \tau'_0, \tau'_1)$  is equal to 0 in  $\mathcal{C}^b\text{add}\mathcal{X}$  if it is null homotopic as a morphism in  $\text{add}\mathcal{X}$ . If  $(V, \tau_0, \tau_1) \in \mathcal{C}^b\text{add}\mathcal{X}$ , then we can define morphisms  $\tau_i : V \rightarrow V$  of degree  $(1 - i, i)$  in such a way that

$$\sum_{i=0}^l \tau_i \tau_{l-i} = 0$$

for any  $l \geq 0$  (see [6, Proposition 2.6]). If  $\alpha : (V, \tau_0, \tau_1) \rightarrow (V', \tau'_0, \tau'_1)$ , then there is a sequence of maps  $\alpha_i : V \rightarrow V'$  of degree  $(-i, i)$  such that  $\alpha_0 = \alpha$  and

$$\sum_{i=0}^l \alpha_i \tau_{l-i} = \sum_{i=0}^l \tau'_i \alpha_{l-i}$$

for any  $l \geq 0$  (see [6, Proposition 2.7]). If  $(V, \tau_0, \tau_1)$  is an object of  $\mathcal{C}^b\text{add}\mathcal{X}$ , then we define  $\text{Tot}(V, \tau_0, \tau_1) \in \mathcal{CA}$  by the formulas

$$\text{Tot}(V, \tau_0, \tau_1)_n = \bigoplus_{i+j=n} V_{i,j}, \quad d_{\text{Tot}(V, \tau_0, \tau_1)} = \sum_{i \geq 0} \tau_i.$$

If  $\alpha : (V, \tau_0, \tau_1) \rightarrow (V', \tau'_0, \tau'_1)$ , then we define  $\text{Tot}\alpha : \text{Tot}(V, \tau_0, \tau_1) \rightarrow \text{Tot}(V', \tau'_0, \tau'_1)$  by the formula

$$\text{Tot}\alpha = \sum_{i \geq 0} \alpha_i.$$

Thus we define a functor  $\text{Tot} : \mathcal{C}^b\text{add}\mathcal{X} \rightarrow \mathcal{K}_p^b\mathcal{A}$  (see [6, Proposition 2.10]). By [6, Proposition 2.11] the functor  $\text{Tot}$  factors through some functor  $\bar{Q} : \mathcal{C}^b\text{add}\mathcal{X} \rightarrow \mathcal{K}_p^b\mathcal{A}$ . We define  $F_\theta$  as the composition

$$\mathcal{K}_p^b\mathcal{B} \xrightarrow{\bar{S}} \mathcal{C}^b\text{add}\mathcal{X} \xrightarrow{\bar{Q}} \mathcal{K}_p^b\mathcal{A}. \quad (3.1)$$

Note that  $F_\theta$  is defined modulo isomorphism. We fix some representative of this equivalence for each tilting functor.

**Proposition 1.** *Let  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b\mathcal{A}$  be a tilting functor and  $X$  be a tilting  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex. If  $\theta_X \cong \theta$ , then  $\mathbf{LT}_X \cong F_\theta$ .*

**Proof** Let  $F_\theta$  be defined by the composition (3.1). Then it is enough to prove that  $T_X|_{\mathcal{K}_p^b\mathcal{B}} \cong \iota \bar{Q} \bar{S}$ , where  $\iota$  is the canonical embedding of  $\mathcal{K}_p^b\mathcal{A}$  to  $\mathcal{KA}$ . We know that there is an isomorphism  $\xi : T_X \mathcal{Y}_\mathcal{B} \cong \iota \theta$ . Let define an isomorphism  $\zeta : T_X|_{\mathcal{K}_p^b\mathcal{B}} \cong \iota \bar{Q} \bar{S}$ . Let  $U = \bigoplus_{i=1}^m \mathcal{B}(-, x_i)$  be a direct sum of representable functors. Then we define

$$\zeta_U : T_X(U) = \bigoplus_{i=1}^m T_X \mathcal{Y}_\mathcal{B}(x_i) \rightarrow \bigoplus_{i=1}^m \iota \theta(x_i) = \iota \bar{Q} \bar{S}(U)$$

by the equality  $\zeta_U = \oplus_{i=1}^m \xi_{x_i}$ . For  $U \in \mathcal{P}_{\mathcal{B}}$  we define

$$\zeta_U = S(\pi_U) \zeta_{W_U} T_X(\iota_U) : T_X(U) \rightarrow S(U) = \iota \bar{Q} \bar{S}(U).$$

It is easy to see that  $\zeta$  defines an isomorphism from  $T_X|_{\mathcal{P}_{\mathcal{B}}}$  to  $\iota \bar{Q} \bar{S}|_{\mathcal{P}_{\mathcal{B}}}$ .

Let now  $U = \oplus_{n \in \mathbb{Z}} U_n$  be an arbitrary object of  $\mathcal{K}_p^b \mathcal{B}$ . Then  $T_X(U)$  is a totalization of a bigraded module  $V = \oplus_{i,j \in \mathbb{Z}} T_X(U_j)_i$  with differential  $\tau_0 = \sum_{j \in \mathbb{Z}} d_{T_X(U_j)}$  of degree  $(1, 0)$  and differential  $\tau_1$  of degree  $(0, 1)$  defined by the equality  $\tau_1|_{V_{i,j}} = (-1)^{i+j} T_X(d_U)|_{T_X(U_j)_i}$ . At the same time  $\iota \bar{Q} \bar{S}(U)$  is a totalization of a bigraded module  $V' = \oplus_{i,j \in \mathbb{Z}} S(U_j)_i$  with some differentials  $\tau'_i$  ( $i \geq 0$ ) of degree  $(1-i, i)$  such that  $\tau'_0 = \sum_{j \in \mathbb{Z}} d_{S(U_j)}$  and  $\tau'_1|_{V_{i,j}} = (-1)^{i+j} S(d_U)|_{S(U_j)_i}$ . Here we write  $S(d_U)$  for some representative of homotopy class of it. If we choose a representative of homotopy class of  $\zeta_{U_j}$  for all  $j \in \mathbb{Z}$ , then we obtain a differential  $\zeta_0 : V \rightarrow V'$  of degree  $(0, 0)$  such that  $\tau'_0 \zeta_0 = \zeta_0 \tau_0$  and  $\tau'_1 \zeta_0$  is homotopic to  $\zeta_0 \tau_1$  if we consider  $V$  and  $V'$  as objects of  $\mathcal{K}_p^b \mathcal{A}$  (i.e. if we forget the grading on  $U$ ). Analogously to [6, Proposition 2.7] we can construct  $\zeta_i$  for  $i \geq 0$  of degree  $(-i, i)$  such that

$$\zeta_l \tau_0 + \zeta_{l-1} \tau_1 = \sum_{i=0}^l \tau'_i \zeta_{l-i}$$

for any  $l \geq 1$ . We define  $\zeta_U = \sum_{i \geq 0} \zeta_i : \text{Tot}(V, \tau_0, \tau_1) \rightarrow \text{Tot}(V', \tau'_0, \tau'_1)$ . If  $U, U' \in \mathcal{K}_p^b \mathcal{B}$  and  $f \in \mathcal{C}^b \mathcal{B}(U, U')$ , then it is clear that  $\zeta_U T_X(f) - \bar{Q} \bar{S}(f) \zeta_U$  equals  $\sum_{i \geq 0} v_i$ , where  $v_i$  is of degree  $(-i, i)$  and  $v_0$  is null homotopic. Then it follows from arguments above that  $\zeta : T_X|_{\mathcal{K}_p^b \mathcal{B}} \rightarrow \iota \bar{Q} \bar{S}$  is a morphism of functors. It is clear that it is actually an isomorphism.  $\square$

Note that by Proposition 1 and Remark 1 an equivalence  $F : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  is standard iff  $F \cong F_\theta$  for some tilting functor  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$ . Moreover,  $F_\theta \cong F_{\theta'}$  iff  $\theta \cong \theta'$ .

## 4 $G$ -functors and orbit categories

We say that  $G$  acts on the category  $\mathcal{A}$  if there is a homomorphism of groups  $\Delta : G \rightarrow \text{Aut}(\mathcal{A})$ . In this case we simply write  $g$  instead of  $\Delta(g)$  for  $g \in G$ . Throughout this section we assume that  $\mathcal{A}$  and  $\mathcal{B}$  are categories with  $G$ -action. Now we recall some definitions from [2].

**Definition 8.** A family  $\eta = (\eta_g)_{g \in G}$  of natural isomorphisms  $\eta_g : F \circ g \rightarrow g \circ F$  is called a  $G$ -equivariance adjuster for the functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  if the diagram

$$\begin{array}{ccc} (F \circ gh)x & \xrightarrow{\eta_{g,h,x}} & (g \circ F \circ h)x \\ & \searrow \eta_{gh,x} & \downarrow g(\eta_{h,x}) \\ & & (gh \circ F)x \end{array}$$

commutes for all  $g, h \in G$  and  $x \in \mathcal{A}$ . We say that  $F$  is a  $G$ -equivariant functor if there is a  $G$ -equivariance adjuster for  $F$ . The functor  $F$  is called *strictly  $G$ -equivariant* if  $F \circ g$  equals  $g \circ F$ , i.e. if the  $G$ -equivariance adjuster can be set to be identity.

**Definition 9.** A  $G$ -functor from  $\mathcal{A}'$  to  $\mathcal{A}$  is a pair  $(F, \eta)$ , where  $F$  is a functor from  $\mathcal{A}'$  to  $\mathcal{A}$  and  $\eta$  is a  $G$ -equivariance adjuster for  $F$ . A morphism from  $(F, \eta)$  to  $(F', \eta')$  is a morphism of functors  $\alpha : F \rightarrow F'$  such that the diagram

$$\begin{array}{ccc} Fg & \xrightarrow{\alpha g} & F'g \\ \eta_g \downarrow & & \downarrow \eta'_g \\ gF & \xrightarrow{g\alpha} & gF' \end{array}$$

commutes for any  $g \in G$ . It is clear that a morphism of  $G$ -functors is an isomorphism iff it is an isomorphism of functors. A  $G$ -functor  $(F, \eta)$  is called a  $G$ -equivalence if  $F$  is an equivalence. If  $F$  is strictly  $G$ -equivariant, we simply write  $F$  for the corresponding  $G$ -functor.

If  $(F, \eta) : \mathcal{A}' \rightarrow \mathcal{A}$  and  $(F', \eta') : \mathcal{A}'' \rightarrow \mathcal{A}'$  are  $G$ -functors, then we define their composition by the formula

$$(F, \eta) \circ (F', \eta') = (FF', \eta F(\eta')) : \mathcal{A}'' \rightarrow \mathcal{A},$$

where

$$(\eta F(\eta'))_{g,x} = \eta_{g,F'x} \circ F(\eta'_{g,x}) : FF'gx \rightarrow gFF'x.$$

It is easy to see that the composition defined above is associative (see [7, Lemma 2.8]). Moreover, it respects isomorphisms of  $G$ -functors. If  $(F, \eta) : \mathcal{A}' \rightarrow \mathcal{A}$  is a  $G$ -equivalence and  $\bar{F}$  is an equivalence quasi-inverse to  $F$ , then there is  $\bar{\eta} = (\bar{\eta}_g)_{g \in G}$  ( $\bar{\eta}_g : \bar{F}g \rightarrow g\bar{F}$ ) such that  $(\bar{F}, \bar{\eta}) : \mathcal{A} \rightarrow \mathcal{A}'$  is a  $G$ -equivalence and

$$(F, \eta) \circ (\bar{F}, \bar{\eta}) \cong \text{Id}_{\mathcal{A}} \text{ and } (\bar{F}, \bar{\eta}) \circ (F, \eta) \cong \text{Id}_{\mathcal{A}'}. \quad \square$$

This follows from the proof of [7, Theorem 9.1]. We call this  $G$ -equivalence  $(\bar{F}, \bar{\eta})$  (which is defined by  $(F, \eta)$  modulo isomorphism of  $G$ -equivalences) the quasi-inverse  $G$ -equivalence to  $(F, \eta)$ .

**Remark 2.** Let  $F, F' : \mathcal{A} \rightarrow \mathcal{A}'$  be functors,  $\eta$  – a  $G$ -equivariance adjuster for  $F$  and  $\xi : F \rightarrow F'$  – an isomorphism. Then  $\eta' = (\eta'_g)_{g \in G}$ , where  $\eta'_{g,x} = g(\xi_x) \circ \eta_{g,x} \circ \xi_{gx}^{-1}$ , is a  $G$ -equivariance adjuster for  $F'$ . Moreover,  $(F', \eta') \cong (F, \eta)$ .

**Definition 10.** The orbit category  $\mathcal{A}/G$  is defined as follows.

- The class of objects of  $\mathcal{A}/G$  is equal to that of  $\mathcal{A}$ .
- Let  $x, y \in \mathcal{A}/G$ . The set  $\mathcal{A}/G(x, y)$  consists of  $f = (f_{h,g})_{g,h \in G}$  such that
  - $f_{h,g} \in \mathcal{A}(gx, hy)$ ;
  - the sets  $\{g \in G \mid f_{g,h} \neq 0\}$  and  $\{g \in G \mid f_{h,g} \neq 0\}$  are finite for any  $h \in G$ ;
  - $f_{lh,lg} = l(f_{h,g})$  for all  $g, h, l \in G$ .
- The composition in  $\mathcal{A}/G$  is defined by the formula

$$(f'_{h,g})_{g,h \in G} (f_{h,g})_{g,h \in G} = \left( \sum_{l \in G} f'_{h,l} f_{l,g} \right)_{g,h \in G}.$$



We can define the action of  $G$  on the category  $\text{Mod}\mathcal{A}$  by the formula  ${}^gX := X \circ g^{-1}$  for  $X \in \text{Mod}\mathcal{A}$  and in the obvious way for morphisms. Note that  ${}^g\mathcal{A}(-, x) \cong \mathcal{A}(-, g(x))$ . This action of  $G$  induces an action of  $G$  on the category  $\mathcal{K}_p^b\mathcal{A}$ . Let  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b\mathcal{A}$  be a tilting functor. If  $\theta$  is  $G$ -equivariant, then the categories  $\mathcal{A}/G$  and  $\mathcal{B}/G$  are derived equivalent by [2, Theorem 4.11].

**Remark 3.** Let  $X$  be a tilting  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$ -complex. The category  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$  can be equipped with the diagonal action of  $G$ , i.e. for  $g \in G$  we put  $g(x, y) = (gx, gy)$  and  $g(f \otimes f') = gf \otimes gf'$  for  $x \in \mathcal{A}$ ,  $y \in \mathcal{B}$  and morphisms  $f$  in  $\mathcal{A}$ ,  $f'$  in  $\mathcal{B}$ . Then it is easy to see that a  $G$ -equivariance adjuster for  $\theta_X : \mathcal{B} \rightarrow \mathcal{K}_p^b\mathcal{A}$  is the same thing as a family of maps  $\phi_g \in \mathcal{C}(\mathcal{A} \otimes \mathcal{B}^{\text{op}})(X, {}^gX)$  such that the diagram

$$\begin{array}{ccc} X^y & \xrightarrow{(\phi_g)_y} & ({}^gX)^y \\ & \searrow (\phi_{gh})_y & \downarrow ({}^g\phi_h)_y \\ & & ({}^{gh}X)^y \end{array}$$

commutes in  $\mathcal{KA}$  for all  $g, h \in G$  and  $y \in \mathcal{B}$ .

## 5 Standard $G$ -equivalences

From here on we equip the category  $\mathcal{A}/G$  with the trivial action of  $G$  (i.e.  $\Delta(g) = \text{Id}_{\mathcal{A}/G}$  for all  $g \in G$ ) for any category  $\mathcal{A}$  with a  $G$ -action. We equip the category  $\mathcal{K}_p^b(\mathcal{A}/G)$  with the trivial action of  $G$  as well.

**Definition 11.** The *canonical functor*  $P : \mathcal{A} \rightarrow \mathcal{A}/G$  is defined by  $P(x) = x$  and  $P(f) = (\delta_{g,h}g(f))_{g,h \in G}$  for  $x, y \in \mathcal{A}$  and  $f \in \mathcal{A}(x, y)$ . Let  $s = (s_g)_{g \in G}$  be the collection of maps  $s_g : Pg \rightarrow P$ , where  $s_{g,x} = (\delta_{hg,h'}\text{Id}_{h'x})_{h,h' \in G} : Pgx \rightarrow Px$ . Then  $s$  is a  $G$ -equivariance adjuster for  $P$ . We call  $(P, s)$  the *canonical  $G$ -functor*.

By [2, Proposition 2.6] every morphism in  $f \in \mathcal{A}/G(x, y)$  can be uniquely presented in the form

$$f = \sum_{g \in G} s_{g,y} \circ Pf_g \quad (5.1)$$

for some  $f_g \in \mathcal{A}(x, gy)$ .

**Definition 12.** We define the *pullup functor*  $P^\bullet : \text{Mod}\mathcal{A}/G \rightarrow \text{Mod}\mathcal{A}$  by the formula  $P^\bullet(X) = X \circ P$  for all  $X \in \text{Mod}\mathcal{A}/G$ . The *pushdown functor*  $P_\bullet : \text{Mod}\mathcal{A} \rightarrow \text{Mod}\mathcal{A}/G$  is the functor left adjoint to  $P^\bullet$ . It also induces a functor  $P_\bullet : \mathcal{K}_p^b\mathcal{A} \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$ .

We will use the explicit description of  $P_\bullet$  obtained in [2, Theorem 4.3]. In particular, we have  $(P_\bullet X)(x) = \bigoplus_{g \in G} X(gx)$ . The same theorem says that the map  $s_\bullet$  defined by the commutative diagram

$$\begin{array}{ccc} (P_\bullet {}^gX)(x) & \xrightarrow{s_{\bullet,g,X,x}} & (P_\bullet X)(x) \\ \parallel & & \parallel \\ \bigoplus_{h \in G} X(g^{-1}hx) & \xrightarrow{(\delta_{g^{-1}h,h'}\text{Id}_{X(h'x)})_{h,h' \in G}} & \bigoplus_{h' \in G} X(h'x) \end{array}$$

is a  $G$ -equivariance adjuster for  $P_\bullet$ . Moreover, by [2, Theorem 4.4] every morphism  $f \in \mathcal{K}_p^b(\mathcal{A}/G)(P_\bullet X, P_\bullet Y)$  can be uniquely presented in the form

$$f = \sum_{g \in G} s_{\bullet, g, Y} \circ P_\bullet f_g \quad (5.2)$$

for some  $f_g \in \mathcal{K}_p^b \mathcal{A}(X, {}^g Y)$ . In addition, we have an isomorphism  $\gamma_y : \mathcal{B}/G(-, Py) \rightarrow P_\bullet \mathcal{B}(-, y)$  ( $y \in \mathcal{B}$ ) defined by the formula  $\gamma_y(s_{g, y} P f) = g^{-1}(f) \in \mathcal{B}(g^{-1}x, y)$  for  $f \in \mathcal{B}(x, gy)$ .

Note also that the Yoneda embedding  $\mathcal{Y}_\mathcal{A} : \mathcal{A} \rightarrow \mathcal{K}_p^b \mathcal{A}$  admits a  $G$ -equivariance adjuster  $\phi = (\phi_g)_{g \in G}$  defined by the formula  $\phi_{g, x}(f) = g^{-1}(f)$  for  $f \in \mathcal{A}(y, gx)$ .

**Lemma 1.** *For all  $g \in G$ ,  $x, y \in \mathcal{A}$  and  $f \in \mathcal{A}(x, gy)$  the following equality holds*

$$\gamma_y^{-1} s_{\bullet, g, \mathcal{A}(-, y)} P_\bullet \phi_{g, y} P_\bullet \mathcal{Y}_\mathcal{A}(f) \gamma_x = \mathcal{Y}_{\mathcal{A}/G}(s_{g, y} P f). \quad (5.3)$$

**Proof** It is enough to prove that the left and the right parts of the equality (5.3) send the element  $s_{h, x} P f' \in \mathcal{A}/G(Pz, Px)$  to the same element of  $\mathcal{A}/G(Pz, Py)$  for all  $h \in G$ ,  $z \in \mathcal{A}$  and  $f' \in \mathcal{A}(z, hx)$ . Direct calculations show that both parts of (5.3) send  $s_{h, x} P f'$  to  $s_{hg, y} P(h(f)f')$ . □

**Definition 13.** A  $G$ -equivalence  $(F, \eta) : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  is called *standard  $G$ -equivalence* if  $F$  is a standard equivalence and there is a standard equivalence  $F' : \mathcal{K}_p^b(\mathcal{B}/G) \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$  such that there is an isomorphism of  $G$ -functors

$$(P_\bullet, s_\bullet) \circ (F, \eta) \cong F'(P_\bullet, s_\bullet). \quad (5.4)$$

We denote by  $\text{TrPic}_G(\mathcal{A}, \mathcal{B})$  the set of isomorphism classes of standard  $G$ -equivalences from  $\mathcal{K}_p^b \mathcal{B}$  to  $\mathcal{K}_p^b \mathcal{A}$ .

It is clear that the composition of standard  $G$ -equivalences and the quasi-inverse  $G$ -equivalence to a standard  $G$ -equivalence are standard.

**Definition 14.** The *derived Picard  $G$ -group* of  $\mathcal{A}$  is the set

$$\text{TrPic}_G(\mathcal{A}) := \text{TrPic}_G(\mathcal{A}, \mathcal{A})$$

with the operation of composition. It follows from the arguments above that it is actually a group.

Let  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a tilting functor and  $\psi$  be a  $G$ -equivariance adjuster for  $\theta$ . We denote  $\theta(\mathcal{B})$  by  $\mathcal{X}$ , note that by agreement  $\theta(\mathcal{B})$  is closed under isomorphism. Then the category  $P_\bullet \mathcal{X}$  is a tilting subcategory for  $\mathcal{A}/G$  (see the proof of [2, Theorem 4.7]). We will construct a tilting functor  $\mu_{\theta, \psi} : \mathcal{B}/G \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$  which induces an equivalence from  $\mathcal{B}/G$  to  $P_\bullet \mathcal{X}$ . We define it on objects by the formula  $\mu_{\theta, \psi}(Py) = P_\bullet \theta(y)$  for  $y \in \mathcal{B}$ . Let us consider a morphism  $f = \sum_{g \in G} s_{g, y} P f_g \in \mathcal{B}/G(Px, Py)$ . Define

$$\mu_{\theta, \psi}(f) = \sum_{g \in G} s_{\bullet, g, \theta(y)} \circ P_\bullet \psi_{g, y} \circ P_\bullet \theta(f_g).$$

It is easy to check that  $\mu_{\theta, \psi}$  is a functor. Since  $\theta$  induces an equivalence to  $\mathcal{X}$ ,  $\mu_{\theta, \psi}$  induces an equivalence to  $P_\bullet \mathcal{X}$  by the arguments above. So an equivalence  $F_{\mu_{\theta, \psi}} : \mathcal{K}_p^b(\mathcal{B}/G) \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$  is defined. Note that if  $(\theta, \psi) \cong (\theta', \psi')$ , then  $\mu_{\theta, \psi} \cong \mu_{\theta', \psi'}$ .

**Proposition 2.** *There is a  $G$ -equivariance adjuster  $\eta$  for  $F_\theta$  such that*

$$F_{\mu_{\theta,\psi}} \circ (P_\bullet, s_\bullet) \cong (P_\bullet, s_\bullet) \circ (F_\theta, \eta).$$

**Proof** Note that  ${}^gU$  lies in  $\mathcal{X}$  for all  $g \in G$  and  $U \in \mathcal{X}$ . Indeed, since  $\theta$  induces an equivalence to  $\mathcal{X}$ , there is some  $y \in \mathcal{B}$  such that  $\theta(y) \cong U$ . Then  ${}^gU \cong {}^g\theta(y) \cong \theta(gy)$  and  ${}^gU$  lies in  $\mathcal{X}$  because  $\mathcal{X}$  is closed under isomorphisms. Then it is easy to see that the action of  $G$  on  $\mathcal{K}_p^b \mathcal{A}$  induces an action on  $\mathcal{K}^b \text{add} \mathcal{X}$  and the  $G$ -functor  $(P_\bullet, s_\bullet) : \mathcal{K}_p^b \mathcal{A} \rightarrow \mathcal{K}_p^b (\mathcal{A}/G)$  induces a  $G$ -functor  $(P_\bullet, s_\bullet) : \mathcal{K}^b \text{add} \mathcal{X} \rightarrow \mathcal{K}^b \text{add} P_\bullet \mathcal{X}$  (here we equip the category  $\mathcal{K}^b \text{add} P_\bullet \mathcal{X}$  with the trivial action of  $G$ ). Let us consider the diagram

$$\begin{array}{ccccc} \mathcal{K}_p^b \mathcal{B} & \xrightarrow{(\bar{S}, \eta_S)} & \mathcal{K}^b \text{add} \mathcal{X} & \xrightarrow{(\bar{Q}, \eta_Q)} & \mathcal{K}_p^b \mathcal{A} \\ \downarrow (P_\bullet, s_\bullet) & & \downarrow (P_\bullet, s_\bullet) & & \downarrow (P_\bullet, s_\bullet) \\ \mathcal{K}_p^b (\mathcal{B}/G) & \xrightarrow{\bar{S}_G} & \mathcal{K}^b \text{add} P_\bullet \mathcal{X} & \xrightarrow{\bar{Q}_G} & \mathcal{K}_p^b (\mathcal{A}/G) \end{array} \quad (5.5)$$

where the rows are the compositions corresponding to (3.1) from the construction of  $F_\theta$  and  $F_{\mu_{\theta,\psi}}$  (if we omit the  $G$ -equivariance adjusters in the upper row). It is enough to show that  $\eta_S$  and  $\eta_Q$  can be constructed in such a way that the diagram (5.5) becomes commutative modulo isomorphism as a diagram of  $G$ -functors. Here we consider the functors in the lower row as strict  $G$ -functors.

It is clear that  $\mathcal{K}_p^b \mathcal{B} = \mathcal{K}^b \text{add} \mathcal{Y}_\mathcal{B}(\mathcal{B})$ . Let us define a  $G$ -equivariance adjuster  $\eta$  for the functor  $S : \text{add} \mathcal{Y}_\mathcal{B}(\mathcal{B}) = \mathcal{P}_\mathcal{B} \rightarrow \text{add} \mathcal{X}$  (see section 3) in the following way. If  $U = \oplus_{i=1}^m \mathcal{B}(-, x_i)$  is a direct sum of representable functors, then

$$\eta_{g,U} = (\oplus_{i=1}^m \psi_{g,x_i}) S(\oplus_{i=1}^m \phi_{g,x_i}^{-1}) : S({}^gU) \cong \oplus_{i=1}^m \theta(gx_i) \rightarrow \oplus_{i=1}^m {}^g\theta(x_i) = {}^gS(U).$$

Let us now consider an arbitrary  $U \in \mathcal{P}_\mathcal{B}$ . Then we define  $\eta_{g,U}$  by the formula

$$\eta_{g,U} = {}^gS(\pi_U) \eta_{g,W_U} S({}^g\iota_U)$$

(see the construction of the functor  $S$  for notation). Direct calculations involving formula (5.3) show that  $(S, \eta)$  is a  $G$ -functor. Then  $\eta$  induces a  $G$ -equivariance adjuster  $\eta_S$  for  $\bar{S}$  in the obvious way. To prove the commutativity of the first square in (5.5) it is enough to prove that the diagram

$$\begin{array}{ccc} \mathcal{P}_\mathcal{B} & \xrightarrow{(S, \eta)} & \text{add} \mathcal{X} \\ \downarrow (P_\bullet, s_\bullet) & & \downarrow (P_\bullet, s_\bullet) \\ \mathcal{P}_{\mathcal{B}/G} & \xrightarrow{S_G} & \text{add} P_\bullet \mathcal{X} \end{array}$$

commutes modulo isomorphism of  $G$ -functors. Let us construct an isomorphism  $\chi : P_\bullet S \rightarrow S_G P_\bullet$ . If  $U = \oplus_{i=1}^m \mathcal{B}(-, x_i)$  is a direct sum of representable functors, then

$$P_\bullet S(U) = \oplus_{i=1}^m P_\bullet \theta(x_i) = S_G(\oplus_{i=1}^m \mathcal{B}/G(-, Px_i)).$$

We set  $\chi_U = S_G(\oplus_{i=1}^m \gamma_{x_i})$ . For an arbitrary  $U \in \mathcal{P}_\mathcal{B}$  we define  $\chi_U$  by the formula

$$\chi_U = S_G(P_\bullet \pi_U) \chi_{W_U} P_\bullet S(\iota_U).$$

Direct calculations involving formula (5.3) show that  $\chi$  is the required isomorphism of  $G$ -functors.

It remains to check the commutativity of the second square in (5.5). Let us take an object of  $\mathcal{K}^b \text{add} \mathcal{X}$  represented by a triple  $(U, \tau_0, \tau_1)$  (see section 3). Then  $P_\bullet U$  can be represented by the triple  $(P_\bullet U, P_\bullet \tau_0, P_\bullet \tau_1)$ . Suppose that  $\bar{Q}$  sends  $(U, \tau_0, \tau_1)$  to the totalization  $(U, \sum_{i \geq 0} \tau_i)$  and  $\bar{Q}_G$  sends  $(P_\bullet U, P_\bullet \tau_0, P_\bullet \tau_1)$  to the totalization  $(P_\bullet U, \sum_{i \geq 0} v_i)$ . It is clear that  $v_0 = P_\bullet \tau_0$  and that  $v_1$  is homotopic to  $P_\bullet \tau_1$  if we consider  $P_\bullet U$  as an object of  $\mathcal{K}_p^b(\mathcal{A}/G)$ . By the results of section 3 there is a sequence of  $\mathcal{A}/G$ -module morphisms  $\alpha_i : P_\bullet U \rightarrow P_\bullet U$  ( $i \geq 0$ ) such that

- $\alpha_i$  is of degree  $(-i, i)$ ,
- $\alpha_0 = \text{Id}_{P_\bullet U}$ ,
- $\sum_{i=0}^l \alpha_i P_\bullet \tau_{l-i} = \sum_{i=0}^l v_i \alpha_{l-i}$ .

In this case define the isomorphism  $\zeta_U$  from the totalization  $(P_\bullet U, \sum_{i \geq 0} P_\bullet \tau_i)$  to the totalization  $(P_\bullet U, \sum_{i \geq 0} v_i)$  by the formula  $\zeta_U = \sum_{i \geq 0} \alpha_i$ . Let  $U, V \in \mathcal{K}^b \text{add} \mathcal{X}$ ,  $f : U \rightarrow V$  be a morphism in  $\mathcal{C}^b \text{add} \mathcal{X}$ . It is clear that the map  $\zeta_V P_\bullet \bar{Q}(f) - \bar{Q}_G P_\bullet(f) \zeta_U$  is a totalization of a map from  $P_\bullet U$  to  $P_\bullet V$  which have nonzero components only in degrees  $(-i, i)$  for  $i > 0$ . It follows from the results of section 3 that the totalization of such a map is null homotopic. So  $\zeta$  gives an isomorphism from  $P_\bullet \bar{Q}$  to  $\bar{Q}_G P_\bullet$ . It remains to construct a  $G$ -equivariance adjuster  $\eta_Q$  for  $\bar{Q}$  such that the diagram

$$\begin{array}{ccc} P_\bullet \bar{Q}^g U & \xrightarrow{\zeta_{gU}} & \bar{Q}_G P_\bullet^g U \\ s_{\bullet, g, \bar{Q}U} \circ P_\bullet \eta_{Q, g, U} \downarrow & & \bar{Q}_G(s_{\bullet, g, U}) \downarrow \\ P_\bullet \bar{Q}U & \xrightarrow{\zeta_U} & \bar{Q}_G P_\bullet U \end{array}$$

commutes for all  $U \in \mathcal{K}^b \text{add} \mathcal{X}$  and  $g \in G$ . The construction of such isomorphisms  $\eta_{Q, g, U} : \bar{Q}^g U \rightarrow \bar{Q}U$  is analogous to the construction of  $\zeta_U$  and so it is left to the reader.

□

**Corollary 1.** *Let  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a tilting functor. Then the following statements are equivalent:*

- 1) *there is a  $G$ -equivariance adjuster for  $\theta$ ;*
- 2) *there is a  $G$ -equivariance adjuster for  $F_\theta$ ;*
- 3) *there is a  $G$ -equivariance adjuster  $\eta$  for  $F_\theta$  such that  $(F_\theta, \eta)$  is a standard  $G$ -equivalence.*

**Proof** The implication "1)  $\Rightarrow$  3)" follows from Proposition 2. Implications "3)  $\Rightarrow$  2)  $\Rightarrow$  1)" are obvious.

□

Let  $(F_\theta, \eta) : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a  $G$ -equivalence. We define  $\psi_{g, y} : \theta(gy) \rightarrow {}^g \theta(y)$  by the formula

$$\psi_{g, y} = \eta_{g, B(-, y)} F_\theta(\phi_{g, y}).$$

It is clear that  $\psi = (\psi_g)_{g \in G}$  is a  $G$ -equivariance adjuster for  $\theta$ .

**Theorem 2.** *Let  $(F_\theta, \eta) : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a standard  $G$ -equivalence. Then  $F_{\mu_{\theta, \psi}}$  is determined by the condition (5.4) uniquely modulo isomorphism.*

**Proof** Since  $(F_\theta, \eta)$  is a standard  $G$ -equivalence, there is some standard equivalence  $F' : \mathcal{K}_p^b(\mathcal{B}/G) \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$  satisfying the condition (5.4).

It is enough to prove that  $F' \mathcal{Y}_{\mathcal{B}/G} \cong \mu_{\theta, \psi}$ . From (5.4) we have a natural isomorphism  $\xi : F' P_\bullet \rightarrow P_\bullet F_\theta$  such that the diagram

$$\begin{array}{ccc} F' P_\bullet {}^g U & \xrightarrow{\xi {}^g U} & P_\bullet F_\theta {}^g U \\ F'(s_{\bullet, g, U}) \downarrow & s_{\bullet, g, F_\theta U \circ P_\bullet}(\eta_{g, U}) \downarrow & \\ F' P_\bullet U & \xrightarrow{\xi U} & P_\bullet F_\theta U \end{array} \quad (5.6)$$

commutes for any  $U \in \mathcal{K}_p^b \mathcal{B}$ . Let us define  $\zeta_{Py} : F' \mathcal{Y}_{\mathcal{B}/G}(Py) \rightarrow \mu_{\theta, \psi}(Py)$  by the formula

$$\zeta_{Py} = \xi_{\mathcal{B}(-, y)} F' \gamma_y.$$

Then using (5.3), (5.6) and the fact that  $\xi$  is a morphism of functors we get

$$\begin{aligned} \zeta_{Py} F' \mathcal{Y}_{\mathcal{B}/G}(s_{g, y} P f) &= \zeta_{Py} F' (\gamma_y^{-1} s_{\bullet, g, B(-, y)} P_\bullet (\phi_{g, y}) \gamma_{gy} \mathcal{B}/G(-, P f)) \\ &= s_{\bullet, g, \theta(y)} \circ P_\bullet \eta_{g, \mathcal{B}(-, y)} \circ \xi_{g \mathcal{B}(-, y)} F' P_\bullet (\phi_{g, y} \mathcal{B}(-, f)) F' (\gamma_x) = \mu_{\theta, \psi}(s_{g, y} P f) \zeta_{Px}. \end{aligned}$$

for all  $x, y \in \mathcal{B}$ ,  $g \in G$ ,  $f \in \mathcal{B}(x, gy)$ . Since any morphism in  $\mathcal{B}/G$  is of the form (5.1),  $\zeta$  is the required isomorphism from  $F' \mathcal{Y}_{\mathcal{B}/G}$  to  $\mu_{\theta, \psi}$ . □

## 6 The maps $\Phi$ and $\Psi$

In this section we define two maps:

$$\Psi_{\mathcal{A}, \mathcal{B}} : \text{TrPic}_G(\mathcal{A}, \mathcal{B}) \rightarrow \text{TrPic}(\mathcal{A}/G, \mathcal{B}/G) \text{ and } \Phi_{\mathcal{A}, \mathcal{B}} : \text{TrPic}_G(\mathcal{A}, \mathcal{B}) \rightarrow \text{TrPic}(\mathcal{A}, \mathcal{B}).$$

Then we investigate some of their properties.

Let  $(F, \eta) : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a standard  $G$ -equivalence. Then we define  $\Phi_{\mathcal{A}, \mathcal{B}}$  as follows:

$$\Phi_{\mathcal{A}, \mathcal{B}}(F, \eta) = F$$

and define  $\Psi_{\mathcal{A}, \mathcal{B}}(F, \eta)$  to be the unique standard equivalence  $F'$  satisfying the condition (5.4). The correctness of the definition of  $\Psi_{\mathcal{A}, \mathcal{B}}$  follows from Theorem 2. It is clear that

$$\Phi_{\mathcal{A}, \mathcal{B}'}((F, \eta) \circ (F', \eta')) = \Phi_{\mathcal{A}, \mathcal{B}}(F, \eta) \circ \Phi_{\mathcal{B}, \mathcal{B}'}(F', \eta')$$

and

$$\Psi_{\mathcal{A}, \mathcal{B}'}((F, \eta) \circ (F', \eta')) = \Psi_{\mathcal{A}, \mathcal{B}}(F, \eta) \circ \Psi_{\mathcal{B}, \mathcal{B}'}(F', \eta').$$

In particular,  $\Phi_{\mathcal{A}} := \Phi_{\mathcal{A}, \mathcal{A}}$  and  $\Psi_{\mathcal{A}} := \Psi_{\mathcal{A}, \mathcal{A}}$  are homomorphisms of groups.

**Definition 15.** A standard equivalence  $F : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  is called a *Morita equivalence* if  $F \cong F_\theta$  where  $\theta(y)$  is isomorphic to some object  $U$  concentrated in degree 0 ( $U_n = 0$  for  $n \neq 0$ ) for any  $y \in \mathcal{B}$ . We denote by  $\text{Pic}(\mathcal{A}, \mathcal{B})$  the set of Morita equivalences from  $\mathcal{B}$  to  $\mathcal{A}$  modulo isomorphisms. It is clear that the composition of Morita equivalences and the inverse to a Morita equivalence are again Morita equivalences. In particular, the set  $\text{Pic}(\mathcal{A}) := \text{Pic}(\mathcal{A}, \mathcal{A})$  is a subgroup of  $\text{TrPic}(\mathcal{A})$ . This group is called the *Picard group* of  $\mathcal{A}$ .

**Theorem 3.** Let  $(F, \eta) : \mathcal{K}_p^b \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a standard  $G$ -equivalence. Then

$$\Phi_{\mathcal{A}, \mathcal{B}}(F, \eta) \in \text{Pic}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \Psi_{\mathcal{A}, \mathcal{B}}(F, \eta) \in \text{Pic}(\mathcal{A}/G, \mathcal{B}/G).$$

In particular,

$$\Phi_{\mathcal{A}}^{-1}(\text{Pic}(\mathcal{A})) = \Psi_{\mathcal{A}}^{-1}(\text{Pic}(\mathcal{A}/G)).$$

**Proof** By Remark 2 we can assume that  $F = F_\theta$  for some tilting functor  $\theta$ . Denote  $\theta(\mathcal{B})$  by  $\mathcal{X}$ . By Theorem 2 we have  $\Psi_{\mathcal{A}, \mathcal{B}}(F, \eta) \cong F_\mu$  for some equivalence  $\mu : \mathcal{B}/G \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$  such that  $\mu(Py) = P_\bullet \theta(y)$  for any  $y \in \mathcal{B}$ .

Suppose that  $F \in \text{Pic}(\mathcal{A}, \mathcal{B})$ . Let us consider  $y \in \mathcal{B}$ . There is some object  $U \in \mathcal{K}_p^b \mathcal{A}$  concentrated in degree 0 such that  $\theta(y) \cong U$ . Then  $\mu(Py) = P_\bullet \theta(y) \cong P_\bullet U$ . It is clear that  $P_\bullet U$  is concentrated in degree 0. Consequently,  $\Psi_{\mathcal{A}, \mathcal{B}}(F, \eta) \in \text{Pic}(\mathcal{A}/G, \mathcal{B}/G)$ .

Suppose now that  $\Psi_{\mathcal{A}, \mathcal{B}}(F, \eta) \in \text{Pic}(\mathcal{A}/G, \mathcal{B}/G)$ . It is enough to prove that any object of  $\mathcal{X}$  is isomorphic in  $\mathcal{K}_p^b \mathcal{A}$  to some object concentrated in degree 0. Any object of  $P_\bullet \mathcal{X}$  is isomorphic in  $\mathcal{K}_p^b(\mathcal{A}/G)$  to some object concentrated in degree 0 by our assumption. Consider some  $U \in \mathcal{X}$ . We know that  $P_\bullet U$  is isomorphic to an object concentrated in degree 0. Then  $P^\bullet P_\bullet U$  is isomorphic to an object concentrated in degree 0 in  $\mathcal{K}_p \mathcal{A}$ . Since  $U$  is a direct summand of  $P^\bullet P_\bullet U$  (see the proofs of [2, Theorems 4.3 and 4.4]),  $U$  is isomorphic to some object concentrated in degree 0. □

**Definition 16.** The *center* of a category  $\mathcal{A}$  is the set of natural transformations from  $\text{Id}_{\mathcal{A}}$  to itself. We denote the center of a category  $\mathcal{A}$  by  $Z(\mathcal{A})$ . By  $Z(\mathcal{A})^*$  we denote the subset of  $Z(\mathcal{A})$  formed by natural isomorphisms. If  $\theta : \mathcal{A} \rightarrow \mathcal{B}$  is a functor, then  $\alpha \in Z(\mathcal{A})$  determines a natural transformation  $\theta(\alpha) : \theta \rightarrow \theta$  by the formula  $\theta(\alpha)_x = \theta(\alpha_x)$ . It is clear that if  $\theta$  is an equivalence, then any natural isomorphism from  $\theta$  to  $\theta$  is of the form  $\theta(\alpha)$  ( $\alpha \in Z(\mathcal{A})^*$ ).

Now let  $F_\theta$  be an element of  $\text{TrPic}(\mathcal{A}, \mathcal{B})$ . We want to determine when  $F_\theta$  lies in the image of  $\Phi_{\mathcal{A}, \mathcal{B}}$ . Let the group  $G$  be given by generators and relations  $G = \langle \{a\}_{a \in A} | \{b\}_{b \in B} \rangle$ . We know from Proposition 2 that  $F_\theta \in \text{Im}(\Phi_{\mathcal{A}, \mathcal{B}})$  iff there is a  $G$ -equivariance adjuster for  $\theta$ . In particular, if  $F_\theta \in \text{Im}(\Phi_{\mathcal{A}, \mathcal{B}})$ , then there is some natural isomorphism  $\varphi_a : \theta a \rightarrow a \theta$  for any  $a \in A$ .

Define  $\varphi_{a^{-1}} : \theta a^{-1} \rightarrow a^{-1} \theta$  by the formula

$$\varphi_{a^{-1}, y} = {}^{a^{-1}}(\varphi_{a, a^{-1}y}^{-1}).$$

Denote  $\tilde{A} := A \cup \{a^{-1} | a \in A\}$ . Let us define natural isomorphisms  $\varphi_{a_1, \dots, a_n} : \theta a_1 \dots a_n \rightarrow a_1 \dots a_n \theta$  for all families  $a_1, \dots, a_n \in \tilde{A}$ . We have done this for the case  $n = 1$ . Let  $\varphi_{a_1, \dots, a_{n-1}} : \theta a_1 \dots a_{n-1} \rightarrow a_1 \dots a_{n-1} \theta$  be defined. Then we define  $\varphi_{a_1, \dots, a_n}$  by the formula

$$\varphi_{a_1, \dots, a_n, x} = {}^{a_1 \dots a_{n-1}} \varphi_{a_n, x} \varphi_{a_1, \dots, a_{n-1}, a_n x}.$$

Let  $a_1, \dots, a_n \in \tilde{A}$  be such elements that  $a_1 \dots a_n \in B$ . Then  $\varphi_{a_1, \dots, a_n}$  is a natural isomorphism from  $\theta$  to itself. So there is a family  $\alpha = (\alpha_b)_{b \in B}$  of elements of  $Z(\mathcal{B})^*$  such that  $\varphi_{a_1, \dots, a_n} = \theta(\alpha_b)$  for  $b = a_1 \dots a_n \in B$ . As it was mentioned above any  $\sigma \in \text{Aut } \mathcal{B}$  induces an automorphism  $\sigma \in \text{Aut}(\mathcal{K}_p^b \mathcal{B})$ . It is clear that  $\sigma$  is a standard derived equivalence lying in  $\text{Pic}(\mathcal{B})$ . Let  $\epsilon_a : \sigma a \rightarrow a \sigma$  ( $a \in A$ ) be a family of natural isomorphisms. We define  $\epsilon_{a_1, \dots, a_n} : \sigma a_1 \dots a_n \rightarrow a_1 \dots a_n \sigma$  for  $a_1, \dots, a_n \in \tilde{A}$  analogously to the definition of  $\varphi_{a_1, \dots, a_n}$ . For  $b = a_1 \dots a_n \in B$  we define  $\epsilon_b$  by the formula  $\epsilon_b = \epsilon_{a_1, \dots, a_n}$ .

**Definition 17.** In the above notation the family of isomorphisms  $\varphi = (\varphi_a)_{a \in A}$  is called an *approximate equivariance adjuster* for  $\theta$ . The family  $\alpha$  is called an *equivariance error* for  $\varphi$ . The family  $\epsilon = (\epsilon_a)_{a \in A}$  is called an *equivariance  $\sigma$ -correction* for  $\alpha$  if  $\alpha_{b, \sigma y} \epsilon_{b, y} = \text{Id}_{\sigma y}$  for any  $b \in B$  and  $y \in \mathcal{B}$ .

**Theorem 4.** Suppose that  $G$  is given by generators and relations. Let  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$  be a tilting functor. Suppose that  $\varphi$  is an approximate equivariance adjuster for  $\theta$  with equivariance error  $\alpha$ . Let  $\sigma$  be some automorphism of  $\mathcal{B}$ . Then  $F_\theta \sigma$  lies in the image of  $\Phi_{\mathcal{A}, \mathcal{B}}$  iff there exists an equivariance  $\sigma$ -correction for  $\alpha$ .

**Proof** By Proposition 2  $F_\theta \sigma \in \text{Im } \Phi_{\mathcal{A}, \mathcal{B}}$  iff there is a  $G$ -equivariance adjuster for  $\theta \sigma$ . Suppose that  $\psi$  is a  $G$ -equivariance adjuster for  $\theta \sigma$ . Then direct calculations show that  $\epsilon$  defined by the equalities  $\psi_{a, x} = \varphi_{a, \sigma x} \circ \theta \epsilon_{a, x}$  is an equivariance  $\sigma$ -correction for  $\alpha$ .

Assume now that  $\epsilon$  is an equivariance  $\sigma$ -correction for  $\alpha$ . For  $a_1, \dots, a_n \in \tilde{A}$  we define  $\psi_{a_1 \dots a_n} : \theta \sigma a_1 \dots a_n \rightarrow a_1 \dots a_n \theta \sigma$  by the formula

$$\psi_{a_1 \dots a_n, x} = \varphi_{a_1, \dots, a_n, \sigma x} \circ \theta \epsilon_{a_1, \dots, a_n, x}.$$

The correctness of this definition follows from the fact that  $\epsilon$  is an equivariance  $\sigma$ -correction for  $\alpha$ . It can be verified by direct calculations that  $\psi$  is a  $G$ -equivariance adjuster for  $\theta \sigma$ . □

## 7 $G$ -grading and the image of $\Psi$

In this section we give a description of the image of  $\Psi_{\mathcal{A}, \mathcal{B}}$  in the case where  $G$  is a finite group. We need the finiteness of  $G$  to prove the following lemma.

**Lemma 2.** Let  $G$  be a finite group and  $\mathcal{X}$  be a subcategory of  $\mathcal{K}_p^b \mathcal{A}$  such that for any  $U \in \mathcal{X}$  and any  $g \in G$  there is  $V \in \mathcal{X}$  such that  ${}^g U \cong V$ . Then  $\mathcal{X}$  is a tilting subcategory for  $\mathcal{A}$  iff  $P_\bullet \mathcal{X}$  is a tilting subcategory for  $\mathcal{A}/G$ .

**Proof** For the proof of the fact that  $P_\bullet \mathcal{X}$  is a tilting subcategory for  $\mathcal{A}/G$  if  $\mathcal{X}$  is a tilting subcategory for  $\mathcal{A}$  see the proof of [2, Theorem 4.7].

Now let  $P_\bullet \mathcal{X}$  be a tilting subcategory for  $\mathcal{A}/G$ . Let us prove that  $\mathcal{K}_p^b \mathcal{A}(U, V[i]) = 0$  for  $U, V \in \mathcal{X}$ ,  $i \neq 0$ . Suppose that it is not true. Then there are  $U, V \in \mathcal{X}$  and  $i \neq 0$  such that  $\mathcal{K}_p^b \mathcal{A}(U, V[i]) \neq 0$ . Let  $f$  be a nonzero element of  $\mathcal{K}_p^b \mathcal{A}(U, V[i])$ . Then  $P_\bullet f$  is a nonzero element of  $\mathcal{K}_p^b \mathcal{A}/G(P_\bullet U, P_\bullet V[i])$  and so  $P_\bullet \mathcal{X}$  is not a tilting subcategory for  $\mathcal{A}/G$ . It remains to prove that any representable functor lies in the subcategory  $\text{thick } \mathcal{X}$ . Note that if  $|G| < \infty$ , then  $P^\bullet$  sends finitely generated modules to finitely generated modules

and so it induces a functor  $P^\bullet : \mathcal{K}_p^b(\mathcal{A}/G) \rightarrow \mathcal{K}_p^b\mathcal{A}$ . Let us consider a functor  $\mathcal{A}(-, x)$  ( $x \in \mathcal{A}$ ). By our assumption  $\mathcal{A}(-, Px) \cong P_\bullet(\mathcal{A}(-, x))$  lies in  $\text{thick}P_\bullet\mathcal{X}$ . Let us consider the subcategory  $\text{thick}P^\bullet P_\bullet\mathcal{X}$  of  $\mathcal{K}_p^b\mathcal{A}$ . It contains the subcategory  $P^\bullet\text{thick}P_\bullet\mathcal{X}$  and so contains  $P^\bullet P_\bullet(\mathcal{A}(-, x)) \cong \bigoplus_{g \in G} \mathcal{A}(-, gx)$ . Since  $\text{thick}P^\bullet P_\bullet\mathcal{X}$  is closed under direct summands it contains  $\mathcal{A}(-, x)$ . It remains to prove that  $\text{thick}\mathcal{X}$  contains  $P^\bullet P_\bullet U$  for any  $U \in \mathcal{X}$ . But  $\text{thick}\mathcal{X}$  contains  ${}^g U$  for any  $U \in \mathcal{X}$  and any  $g \in G$ . So it contains  $P^\bullet P_\bullet U \cong \bigoplus_{g \in G} {}^g U$  for any  $U \in \mathcal{X}$ . □

**Definition 18.** A  $G$ -graded category is a category  $\mathcal{A}$  having a family of direct sum decompositions  $\mathcal{A}(x, y) = \bigoplus_{g \in G} \mathcal{A}(x, y)^{(g)}$  ( $x, y \in \mathcal{A}$ ) of  $\mathbf{k}$ -modules such that the composition of morphisms gives the inclusions  $\mathcal{A}(y, z)^{(g)} \mathcal{A}(x, y)^{(h)} \subset \mathcal{A}(x, z)^{(gh)}$  for all  $x, y, z \in \mathcal{A}$  and  $g, h \in G$ . If  $f \in \mathcal{A}(x, y)^{(g)}$ , then we say that  $f$  is of  $G$ -degree  $g$ . A functor  $F : \mathcal{A} \rightarrow \mathcal{A}'$  between  $G$ -graded categories is called *degree-preserving* if  $F(\mathcal{A}(x, y)^{(g)}) \subset \mathcal{A}'(Fx, Fy)^{(g)}$  for all  $x, y \in \mathcal{A}$  and  $g \in G$ .

By [2, Lemma 5.4] there is a  $G$ -grading on  $\mathcal{B}/G$  such that  $Pf$  is of degree  $1_G$  and  $s_{g,x}$  is of degree  $g^{-1}$ .

**Definition 19.** Let  $\mathcal{A}$  be a  $G$ -graded category. A  $G$ -graded  $\mathcal{A}$ -complex is an  $\mathcal{A}$ -complex  $U$  with a family of direct sum decompositions  $U(x) = \bigoplus_{g \in G} U(x)^{(g)}$  ( $x \in \mathcal{A}$ ) such that  $d_{U,x}(U(x)^{(g)}) \subset U(x)^{(g)}$  and  $U(f)(U(x)^{(g)}) \subset U(y)^{(gh)}$  for all  $x, y \in \mathcal{A}$ ,  $g, h \in G$ ,  $f \in \mathcal{A}(y, x)^{(h)}$ . If  $U, V$  are  $G$ -graded complexes, then we say that  $f : U \rightarrow V$  is of degree  $h$  if  $f_x(U(x)^{(g)}) \subset V(x)^{(hg)}$ . Let us now define a category  $\mathcal{K}_{p,G}^b\mathcal{A}$ . It's objects are  $G$ -graded  $\mathcal{A}$ -complexes  $U$  which lie in  $\mathcal{K}_p^b\mathcal{A}$  considered as complexes without grading. If  $U, V \in \mathcal{K}_{p,G}^b\mathcal{A}$ , then  $\mathcal{K}_{p,G}^b\mathcal{A}(U, V) = \mathcal{K}_p^b\mathcal{A}(U, V)$ .

Let  $\mathcal{A}$  be a  $G$ -graded category and  $U, V \in \mathcal{K}_{p,G}^b\mathcal{A}$ . Then we denote by  $\mathcal{K}_{p,G}^b\mathcal{A}(U, V)^{(g)}$  the set of morphisms in  $\mathcal{K}_p^b\mathcal{A}(U, V)$  which can be presented by a morphism of degree  $g$  in  $\mathcal{CA}(U, V)$ . It is not hard to check that  $\mathcal{K}_{p,G}^b\mathcal{A}(x, y) = \bigoplus_{g \in G} \mathcal{K}_{p,G}^b\mathcal{A}(U, V)^{(g)}$  and this decomposition turns  $\mathcal{K}_{p,G}^b\mathcal{A}$  into a  $G$ -graded category. For  $U \in \mathcal{K}_{p,G}^b\mathcal{A}$  we denote by  $\bar{U}$  the corresponding object of  $\mathcal{K}_p^b\mathcal{A}$ . Note that any equivalence  $\theta : \mathcal{B} \rightarrow \mathcal{K}_{p,G}^b\mathcal{A}$  determines an equivalence  $\bar{\theta} : \mathcal{B} \rightarrow \mathcal{K}_p^b\mathcal{A}$  in an obvious way.

**Theorem 5.** Suppose that  $G$  is a finite group. Let  $F : \mathcal{K}_p^b(\mathcal{B}/G) \rightarrow \mathcal{K}_p^b(\mathcal{A}/G)$  be a standard equivalence. Then  $F$  lies in the image of  $\Psi_{\mathcal{A},\mathcal{B}}$  iff there is a degree-preserving functor  $\mu : \mathcal{B}/G \rightarrow \mathcal{K}_{p,G}^b(\mathcal{A}/G)$  such that  $\bar{\mu}$  is a tilting functor and  $F \cong F_{\bar{\mu}}$ .

**Proof** If  $F$  lies in the image of  $\Psi_{\mathcal{A},\mathcal{B}}$ , then it is isomorphic to  $F_{\mu_{\theta,\psi}}$  for some tilting functor  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b\mathcal{A}$  and a  $G$ -equivariance adjuster  $\psi$  for  $\theta$ . Let us define a  $G$ -grading on  $\mu_{\theta,\psi}(Py) = P_\bullet\theta(y)$  ( $y \in \mathcal{B}$ ) as follows:  $(P_\bullet\theta(y))(Px)^{(g)} = \theta(y)(gx)$ . Then it can be easily verified that  $\mu_{\theta,\psi}$  defines a degree-preserving functor from  $\mathcal{B}/G$  to  $\mathcal{K}_{p,G}^b(\mathcal{A}/G)$ . Note that this part of the prove does not require the finiteness of  $G$ .

Now let  $\mu : \mathcal{B}/G \rightarrow \mathcal{K}_{p,G}^b(\mathcal{A}/G)$  be a degree-preserving functor such that  $\bar{\mu}$  is a tilting functor. Let us prove that  $F_{\bar{\mu}}$  lies in the image of  $\Psi_{\mathcal{A},\mathcal{B}}$ . By Proposition 2 it is enough to find a tilting functor  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b\mathcal{A}$  and a  $G$ -equivariance adjuster  $\psi$  for  $\theta$  such that  $F_{\bar{\mu}} \cong F_{\mu_{\theta,\psi}}$ .



For  $U \in \mathcal{K}_{p,G}^b(\mathcal{A}/G)$  we define  $\tilde{U} \in \mathcal{K}_p^b \mathcal{A}$  in the following way. It is defined on objects by the formula  $\tilde{U}(x) = U(Px)^{(1)}$  and on morphisms by the formula  $\tilde{U}(f) = U(Pf)|_{U(x)^{(1)}}$  ( $f \in \mathcal{A}(x, y)$ ). The differential  $d_{\tilde{U}}$  is defined by the formula  $d_{\tilde{U},x} = d_{U,Px}|_{U(Px)^{(1)}}$ . The correctness of this definition follows from the definition of a  $G$ -graded complex. Also we define a morphism  $\xi_U : \tilde{U} \rightarrow P_\bullet \tilde{U}$  as follows:

$$\xi_{U,x} = \bigoplus_{g \in G} U(s_{g,x})|_{U(Px)^{(g)}} : \bigoplus_{g \in G} U(Px)^{(g)} \rightarrow \bigoplus_{g \in G} U(Pgx)^{(1)}.$$

Let us now define  $\theta : \mathcal{B} \rightarrow \mathcal{K}_p^b \mathcal{A}$ . We define it on objects by  $\theta(y) = \widetilde{\mu(Py)}$ . For  $f \in \mathcal{B}(y, z)$  we define the natural transformation  $\theta(f)$  by the formula

$$\theta(f)_x = \mu(Pf)_{Px}|_{\mu(Py)(Px)^{(1)}} : \theta(y)(x) \rightarrow \theta(z)(x)$$

for all  $x \in \mathcal{A}$ . Let  $\psi_{g,y,x} : \theta(gy)(x) \rightarrow {}^g\theta(y)(x)$  ( $x \in \mathcal{A}$ ,  $y \in \mathcal{B}$ ,  $g \in G$ ) be the composition

$$\theta(gy)(x) = \mu(Pgy)(Px)^{(1)} \xrightarrow{\mu(s_{g,y})^x} \mu(Py)(Px)^{(g^{-1})} \xrightarrow{\mu(y)(s_{g^{-1}}^{-1},x)} \mu(Py)(Pg^{-1}x)^{(1)} = {}^g\theta(y)(x).$$

It is not hard to prove that the following conditions hold:

- 1)  $\xi_U$  is an isomorphism in  $\mathcal{K}_p^b(\mathcal{A}/G)$ ;
- 2)  $\theta$  induces an equivalence from  $\mathcal{B}$  to  $\theta(\mathcal{B})$ ;
- 3)  $\psi$  is a  $G$ -equivariance adjuster for  $\theta$ ;
- 4) The family of morphisms  $\zeta_{Py} = \xi_{\mu(Py)} : \overline{\mu(Py)} \rightarrow P_\bullet \theta(y) = \mu_{\theta,\psi}(Py)$  defines an isomorphism from  $\bar{\mu}$  to  $\mu_{\theta,\psi}$ .

It follows from 1)–3) and Lemma 2 that  $\theta(\mathcal{B})$  is a tilting subcategory for  $\mathcal{A}$ , hence the theorem is proved. □

## 8 The action of the Nakayama automorphism on Frobenius algebras

From here on we assume that  $\mathbf{k}$  is a field. Let  $R$  be an associative finite dimensional  $\mathbf{k}$ -algebra. It can be considered as a category with one object and so the results of the previous sections can be applied to  $R$ . We denote the unique object of  $R$  by  $e$ . We apply these results in the case where  $R$  is a Frobenius algebra and  $G$  is a finite cyclic group which acts on  $R$  by powers of a Nakayama automorphism. First, let us recall the definition of a Frobenius algebra.

**Definition 20.** An algebra  $R$  is called *Frobenius* if there is a linear map  $\epsilon : R \rightarrow k$  such that the bilinear form  $\langle a, b \rangle = \epsilon(ab)$  is nondegenerate. The Nakayama automorphism  $\nu : R \rightarrow R$  is the automorphism which satisfies the equation  $\langle a, b \rangle = \langle b, \nu(a) \rangle$  for all  $a, b \in R$ . If the bilinear form on  $R$  can be chosen in such a way that  $\langle a, b \rangle = \langle b, a \rangle$  for all  $a, b \in R$ , then the algebra  $R$  is called *symmetric*.

From here on we fix some Frobenius algebra  $R$ , and some Nakayama automorphism  $\nu$  of  $R$ . Moreover, we assume that there is an integer  $n > 0$  such that  $\nu^n = \text{Id}_R$ . Then the

cyclic group  $G = \langle g | g^n \rangle$  acts on  $R$  by the following rule:  $\Delta(g) = \nu$ . Note that  $R/G$  is a symmetric algebra. Indeed, if  $\langle , \rangle$  is the bilinear form on  $R$ , then  $\langle , \rangle_G$  defined by the formula

$$\langle s_{g^k}Pa, s_{g^l}Pb \rangle_G = \delta_{g^{k+l+1}, 1_G} \langle \nu^l a, b \rangle$$

is the desired bilinear form on  $R/G$ . So the maps  $\Phi_R$  and  $\Psi_R$  allow us to transfer some information from the derived Picard group of a symmetric algebra to the derived Picard group of a Frobenius algebra.

We have the following application of Theorem 4.

**Proposition 3.** 1) *If for any  $a \in Z(R)^*$  there exists an element  $b \in Z(R)^*$  such that  $ab^n = 1$ , then  $\Phi_R$  is surjective.*  
 2) *If for any  $a \in Z(R)^*$  there exists an element  $b \in R^*$  and an automorphism  $\sigma \in \text{Aut } R$  such that  $b\nu(b) \dots \nu^{n-1}(b) = a$  and  $\sigma\nu\sigma^{-1}(c) = b\nu(c)b^{-1}$  for any  $c \in R$ , then  $\text{Cok } \Phi_R$  is generated by the images of elements from  $\text{Pic}(R)$ .*

**Proof** Note that the functor  $\nu(-) : \mathcal{K}_p^b R \rightarrow \mathcal{K}_p^b R$  is isomorphic to the Nakayama functor. Then by [3, Proposition 5.2] we have an isomorphism  $\eta_F : F \circ \nu \cong \nu \circ F$  for any standard equivalence  $F : \mathcal{K}_p^b R \rightarrow \mathcal{K}_p^b R$ . If  $F$  is given by a tilting functor  $\theta : R \rightarrow \mathcal{K}_p^b R$ , then  $\theta \cong F\mathcal{Y}_R$ . So the isomorphism  $\eta_{F, \mathcal{Y}(e)} \circ F(\phi_{g,e}) : F\mathcal{Y}_R\nu(e) \rightarrow \nu F\mathcal{Y}_R(e)$  gives an isomorphism  $\varphi_{g,e} : \theta\nu(e) \rightarrow \nu\theta(e)$ . Then  $\varphi$  is an approximate equivariance adjuster for  $\theta$ .

1) Note that  $\nu(b) = b$  for any  $b \in Z(R)$ . Then it follows from the assumption that there is an equivariance  $\text{Id}_R$ -correction for any equivariance error. So  $\Phi_R$  is surjective by Theorem 4.

2) It follows from the assumption that for any equivariance error  $a$  there is some  $\sigma \in \text{Aut } R$  such that there exists an equivariance  $\sigma$ -correction for  $a$ . So by Theorem 4 for any  $F \in \text{TrPic}(R)$  there is some  $\sigma \in \text{Aut } R$  such that  $F\sigma$  lies in the image of  $\Phi_R$ . Then the image of  $F$  in  $\text{Cok } \Phi_R$  equals the image of  $\sigma^{-1}$ . So the assertion follows from the fact that  $\sigma^{-1} \in \text{Pic}(R)$ .

□

**Corollary 2.** *If the field  $\mathbf{k}$  is algebraically closed and its characteristic does not divide  $n$ , then  $\Phi_R$  is surjective.*

**Proof** We may assume that  $R$  is an indecomposable algebra. Then any element  $a \in Z(R)^*$  is of the form  $a = \kappa(1 + Q)$  for some  $\kappa \in \mathbf{k}$  and nilpotent  $Q \in Z(R)$ . Since  $\mathbf{k}$  is algebraically closed, there is some  $\bar{\kappa} \in \mathbf{k}$  such that  $\kappa = \bar{\kappa}^n$ . Then  $a = b^n$  for

$$b = \bar{\kappa} \sum_{i \geq 0} \frac{\prod_{j=0}^{i-1} (\frac{1}{n} - j)}{i!} Q^i.$$

□

## 9 Application: generators of the derived Picard group of a self-injective Nakayama algebra

From here on we assume that  $\mathbf{k}$  is an algebraically closed field. In this section we apply the methods of the previous sections to obtain generators of the derived Picard group of algebras  $\mathcal{N}(nm, tm)$  defined in the following way. Let  $m, n, t > 0$  be some integers. We suppose that  $n$  and  $t$  are coprime. Let  $\mathcal{Q}(nm)$  be a cyclic quiver with  $nm$  vertices, i.e. the quiver whose vertex set is  $\mathbb{Z}_{nm}$  and whose arrows are  $\beta_i : i \rightarrow i + 1$  ( $i \in \mathbb{Z}_{nm}$ ). Let  $\mathcal{I}(nm, tm)$  be an ideal in the path algebra of  $\mathcal{Q}(nm)$  generated by all paths of length  $tm + 1$ . We denote  $\mathcal{N}(nm, tm) := \mathbf{k}\mathcal{Q}(nm)/\mathcal{I}(nm, tm)$ . For  $i \in \mathbb{Z}_{nm}$  we denote by  $e_i$  the primitive idempotent corresponding to the vertex  $i$  and by  $P_i$  the projective module  $e_i\mathcal{N}(nm, tm)$ . For a path  $w$  from the vertex  $i$  to the vertex  $j$  we denote by  $w$  the unique homomorphism from  $P_i$  to  $P_j$  which sends  $e_i$  to  $w$  as well. Also we introduce the following auxiliary notation:

$$\beta_{i,k} = \beta_{i+k-1} \cdots \beta_i.$$

It is well-known that  $\mathcal{N}(nm, tm)$  is a Frobenius algebra with a Nakayama automorphism  $\nu$  defined as follows:  $\nu(e_i) = e_{i-tm}$  and  $\nu(\beta_i) = \beta_{i-tm}$ . If  $U$  is a module, then we also denote by  $U$  the corresponding complex concentrated in degree 0. For  $i \in \mathbb{Z}_{nm}$ ,  $1 \leq k \leq m-1$  we introduce the complex

$$X_i := P_{i-tm} \xrightarrow{\beta_{i-tm}} P_{i-tm+1} \xrightarrow{\beta_{i-tm+1,tm}} P_{i+1}$$

concentrated in degrees -2, -1 and 0 and the complexes

$$Y_{i,k} := P_i \xrightarrow{\beta_{i,k}} P_{i+k}$$

concentrated in degrees 0 and 1.

If  $m > 1$ , then for  $0 \leq l \leq m-1$  we introduce the  $\mathcal{N}(nm, tm)$ -complex

$$H_l^{nm} = \left( \bigoplus_{i \in \mathbb{Z}_{nm}, m \nmid i-l} P_i \right) \oplus \left( \bigoplus_{i \in \mathbb{Z}_{nm}, m \mid i-l} X_i \right).$$

In this case we can define an algebra isomorphism

$$\theta_l^{nm} : \mathcal{N}(nm, tm) \rightarrow \mathcal{K}_p^b(\mathcal{N}(nm, tm))(H_l^{nm}, H_l^{nm}).$$

We define it on idempotents by the formula

$$\theta_l^{nm}(e_i) = \begin{cases} \text{Id}_{P_i} & \text{if } m \nmid i-l \text{ and } m \nmid i-1-l, \\ \text{Id}_{P_{i+1}} & \text{if } m \mid i-l, \\ \text{Id}_{X_i} & \text{if } m \mid i-1-l. \end{cases}$$

We define  $\theta_l^{nm}(\beta_i)$  ( $i \in \mathbb{Z}_{nm}$ ) in the obvious way (it equals 0 in all degrees except for the zero degree and equals  $\beta_i$ ,  $\beta_i\beta_{i+1}$  or  $\text{Id}_{P_{i+1}}$  depending on  $i$  in the zero degree).

If  $m > 1$  and  $t = 1$ , then for  $0 \leq l \leq m-1$  we introduce the  $\mathcal{N}(nm, m)$ -complex

$$Q_l^{nm} = \bigoplus_{i \in \mathbb{Z}_{nm}, m \mid i-l} \left( P_i \oplus \bigoplus_{k=1}^{m-1} Y_{i,k} \right).$$

In this case we can define an algebra isomorphism

$$\varepsilon_l^{nm} : \mathcal{N}(nm, m) \rightarrow \mathcal{K}_p^b(\mathcal{N}(nm, m))(Q_l^{nm}, Q_l^{nm}).$$

We define it on idempotents by the fomula

$$\varepsilon_l^{nm}(e_i) = \begin{cases} \text{Id}_{P_i} & \text{if } m \mid i - l, \\ \text{Id}_{Y_{i+k, m-k}} & \text{if } m \mid i + k - l \text{ for some } k, 1 \leq k \leq m - 1. \end{cases}$$

We define  $\varepsilon_l^{nm}(\beta_i)$  ( $i \in \mathbb{Z}_{nm}$ ) in the following way. It equals 0 in all degrees except 0 and 1. In degree 0 it equals  $\text{Id}_{P_{i+k}}$  if  $m \mid i + k - l$  for  $1 \leq k \leq m - 1$  and equals  $\beta_{i, m}$  if  $m \mid i - l$ . In degree 1 it equals  $\beta_{i+m}$  if  $m \nmid i - l$  and  $m \nmid i + 1 - l$  and equals 0 if  $m \mid i - l$  or  $m \mid i + 1 - l$ .

In this section we prove the following theorem.

**Theorem 6.** 1) If  $m = 1$ , then  $\text{TrPic}(\mathcal{N}(n, t))$  is generated by the shift and  $\text{Pic}(\mathcal{N}(n, t))$ ; 2) If  $m > 1$ ,  $t > 1$ , then  $\text{TrPic}(\mathcal{N}(nm, tm))$  is generated by the shift,  $\text{Pic}(\mathcal{N}(nm, tm))$  and  $F_{\theta_l^{nm}}$  ( $l \in \mathbb{Z}_m$ ); 3) If  $m > 1$ ,  $t = 1$ , then  $\text{TrPic}(\mathcal{N}(nm, m))$  is generated by the shift,  $\text{Pic}(\mathcal{N}(nm, m))$ ,  $F_{\theta_l^{nm}}$  and  $F_{\varepsilon_l^{nm}}$  ( $l \in \mathbb{Z}_m$ ).

It is clear that  $\mathcal{N}(nm, tm)$  is symmetric iff  $n = 1$ . If in addition  $m = 1$ , then  $\mathcal{N}(1, t)$  is a local algebra and so  $\text{TrPic}(\mathcal{N}(1, t))$  is generated by the shift and  $\text{Pic}(\mathcal{N}(1, t))$  by the results of [8], [9]. So the first assertion of Theorem 6 holds for  $n = 1$ . The assertions 2) and 3) of the theorem for  $n = 1$  follow from the results of [4], where the set of generators of the derived Picard group was described in the case  $n = 1$ ,  $m > 1$ . Moreover, it was proved there that any element of  $\text{TrPic}(\mathcal{N}(m, tm))$  is of the form  $UV$ , where  $V \in \text{Pic}(\mathcal{N}(m, tm))$  and  $U$  is a product of elements listed in the points 2)–3) except for  $\text{Pic}(\mathcal{N}(m, tm))$ .

Now let us consider  $n > 1$ . Let  $G = \langle g \mid g^n \rangle$  be a cyclic group which acts on  $\mathcal{N}(nm, tm)$  by the rule  $\Delta(g) = \nu$ . It is well-known that  $\mathcal{N}(nm, tm)/G$  is Morita equivalent to  $\mathcal{N}(m, tm)$ . We need the explicit formula for this equivalence to obtain the isomorphism of the derived Picard groups defined by it. Let  $W = \bigoplus_{j \in \mathbb{Z}_n} W_j$  where  $W_j$  is isomorphic to  $\mathcal{N}(m, tm)$  as a right  $\mathcal{N}(m, tm)$ -module. As it was mentioned above every path  $w$  in  $\mathcal{Q}(m)$  defines a homomorphism  $w : W_j \rightarrow W_j$ . Thus, there is a left  $\mathcal{N}(m, tm)$ -module structure on  $W_j$ . Let us define a left  $\mathcal{N}(nm, tm)/G$ -module structure on  $W$ . Let  $s_{i,j} : W_i \rightarrow W_j$  ( $i, j \in \mathbb{Z}_n$ ) be the isomorphism arising from  $\text{Id}_{\mathcal{N}(m, tm)}$ . Let  $i \in \mathbb{Z}_{nm}$  be represented by an integer number  $0 \leq \bar{i} \leq nm - 1$ . Present  $\bar{i}$  in the form  $\bar{i} = \bar{q}m + \bar{r}$ , where  $0 \leq \bar{r} < m$ . Let  $q \in \mathbb{Z}_n$  and  $r \in \mathbb{Z}_m$  be elements represented by  $\bar{q}$  and  $\bar{r}$  respectively. Consider an element  $x \in W$ . Suppose that  $x \in W_j$  for some  $j \in \mathbb{Z}_n$ . Then we define

$$(Pe_i)x = (\delta_{q,j}e_r)x, (P\beta_i)x = (\delta_{q,j}\beta_r)x \text{ and } s_{g^l}x = s_{j, j+ltm}(x).$$

It is clear that in such a way  $W$  becomes a  $\mathcal{N}(nm, tm)/G - \mathcal{N}(m, tm)$ -bimodule which induces a Morita equivalence

$$T_W = - \otimes_{\mathcal{N}(nm, tm)/G} W : \mathcal{K}_p^b(\mathcal{N}(nm, tm)/G) \rightarrow \mathcal{K}_p^b(\mathcal{N}(m, tm)).$$

We define  $L : \text{TrPic}(\mathcal{N}(m, tm)) \rightarrow \text{TrPic}(\mathcal{N}(nm, tm)/G)$  by the formula

$$L(F) = \bar{T}_W \circ F \circ T_W,$$

where  $\bar{T}_W$  is a quasi-inverse equivalence for  $T_W$ . It is clear that  $L$  sends the shift to the shift and  $\text{Pic}(\mathcal{N}(m, tm))$  to  $\text{Pic}(\mathcal{N}(nm, tm)/G)$ .

There are  $G$ -equivariance adjusters  $\psi_l^{nm}$  for  $\theta_l^{nm}$  and  $\varphi_l^{nm}$  for  $\varepsilon_l^{nm}$ . The maps  $\psi_{l, g^p}^{nm}$  and  $\varphi_{l, g^p}^{nm}$  can be constructed in the obvious way as the sums of the isomorphisms of the form

$$P_{i-ptm} \cong {}^{\nu^p}P_i, X_{i-ptm} \cong {}^{\nu^p}X_i \text{ and } Y_{i-ptm, k} \cong {}^{\nu^p}Y_{i, k}.$$

Then we have the following lemma.

**Lemma 3.** 1) If  $m > 1$ , then for  $0 \leq l \leq m-1$  we have  $L(F_{\theta_l^m}) \cong F_{\mu_{\theta_l^{nm}, \psi_l^{nm}}}$ .  
2) If  $m > 1$  and  $t = 1$ , then for  $0 \leq l \leq m-1$  we have  $L(F_{\varepsilon_l^m}) \cong F_{\mu_{\varepsilon_l^{nm}, \varphi_l^{nm}}}$ .

**Proof** 1) We will prove that

$$F_{\theta_l^m} \circ T_W \cong T_W \circ F_{\mu_{\theta_l^{nm}, \psi_l^{nm}}}. \quad (9.1)$$

Let us describe the left part of this equality. Let  $H = \bigoplus_{j \in \mathbb{Z}_n} H_j$ , where  $H_j \cong H_l^m$  as a right  $\mathcal{N}(m, tm)$ -complex. Denote by  $s'_{i,j} : H_i \rightarrow H_j$  ( $i, j \in \mathbb{Z}_n$ ) the isomorphism arising from  $\text{Id}_{H_l^m}$ . In addition, for  $u \in \mathcal{K}_p^b(\mathcal{N}(m, tm))(H_l^m, H_l^m)$  we denote by  $u$  the corresponding morphism from  $H_j$  to  $H_j$ . Let us define  $\theta : \mathcal{N}(nm, tm)/G \rightarrow \mathcal{K}_p^b(\mathcal{N}(m, tm))(H, H)$ . Consider  $i \in \mathbb{Z}_{nm}$ . Let  $q \in \mathbb{Z}_n$  and  $r \in \mathbb{Z}_m$  be as above. Then for  $x \in H_j$  ( $j \in \mathbb{Z}_n$ ) we define

$$\theta(e_i)(x) = \delta_{j,q} \theta_l^m(e_r)(x), \theta(\beta_i)(x) = \delta_{j,q} \theta_l^m(\beta_r)(x) \text{ and } \theta(s_{g^l})(x) = s'_{j, j+ltm}(x).$$

Then the left part of the equality (9.1) gives  $F_\theta$ . It is not hard to construct an isomorphism

$$\xi : H \rightarrow (P_\bullet H_l^{nm}) \otimes_{\mathcal{N}(nm, tm)/G} W$$

such that  $\xi \theta(c) = (\mu_{\theta_l^{nm}, \psi_l^{nm}}(c) \otimes_{\mathcal{N}(nm, tm)/G} \text{Id}_W) \xi$  for any  $c \in \mathcal{N}(nm, tm)/G$ . The existence of such  $\xi$  gives the isomorphism (9.1).

2) The proof is similar and so it is left to the reader. □

Let us now apply the results of the previous sections to the algebra  $\mathcal{N}(nm, tm)$ .

**Lemma 4.** 1) If  $\text{char } \mathbf{k} \nmid n$ , then  $\Phi_{\mathcal{N}(nm, tm)}$  is surjective.  
2) If  $\text{char } \mathbf{k} \mid n$ , then  $\text{Cok } \Phi_{\mathcal{N}(nm, tm)}$  is generated by images of elements from  $\text{Pic}(\mathcal{N}(nm, tm))$ .

**Proof** 1) Follows directly from Corollary 2.

2) Note, that in this case  $n > 1$ . It is enough to prove that the condition of the second part of Proposition 3 is satisfied. Consider  $a \in Z(\mathcal{N}(nm, tm))^*$ . Let us introduce the notation  $u := \sum_{i \in \mathbb{Z}_{nm}} \beta_{i, nm}$ ,  $\bar{u} := \sum_{i=1}^m \beta_{i, nm}$ . It can be easily proved that  $a = \sum_{0 \leq k \leq \frac{t}{n}} c_k u^k$  for some  $c_k \in \mathbf{k}$ ,  $c_0 \neq 0$ . Since  $\mathbf{k}$  is algebraically closed we may assume that  $c_0 = 1$ . Then  $a = b\nu(b) \dots \nu^{n-1}(b)$  for  $b = 1 + \sum_{1 \leq k \leq \frac{t}{n}} c_k \bar{u}^k$ . Let us denote by  $\gamma$  the automorphism of  $\mathcal{N}(nm, tm)$  defined by the formula  $\gamma(x) = b\nu(x)b^{-1}$  for  $x \in \mathcal{N}(nm, tm)$ . It remains to find such  $\sigma \in \text{Aut } \mathcal{N}(nm, tm)$  that  $\sigma^{-1}\nu\sigma = \gamma$ .

The automorphism  $\gamma$  equals  $\nu$  on the idempotents and is defined on the arrows by the formula

$$\gamma(\beta_i) = \begin{cases} \beta_{i-tm}, & \text{if } i \neq tm \text{ and } i \neq (t+1)m, \\ a\beta_0, & \text{if } i = tm, \\ a^{-1}\beta_m, & \text{if } i = (t+1)m. \end{cases}$$

Let  $0 < p < n$  be such a number that  $n \mid pt - 1$ . We define  $\sigma$  in the following way. It is identical on the idempotents and is defined on arrows in such a way that

$$\sigma^{-1}(\beta_i) = \begin{cases} a^{-1}\beta_i, & \text{if } i = (1+k)tm \text{ for some } 0 \leq k < p, \\ \beta_i, & \text{otherwise.} \end{cases}$$

It is easy to verify that  $\sigma^{-1}\nu\sigma = \gamma$ .

□

**Proof of Theorem 6** It was mentioned above that the theorem is true for  $n = 1$ . Consider  $n > 1$ . We want to prove that some set of elements of  $\text{TrPic}(\mathcal{N}(nm, tm))$  generates it. Denote this set by  $M$ . It follows from Lemma 4 that the image of  $\Phi_{\mathcal{N}(nm, tm)}$  and  $\text{Pic}(\mathcal{N}(nm, tm))$  generates  $\text{TrPic}(\mathcal{N}(nm, tm))$ .

Let  $F \in \text{TrPic}_G(\mathcal{N}(nm, tm))$ . It is enough to prove that  $\Phi_{\mathcal{N}(nm, tm)}(F)$  lies in the subgroup of  $\text{TrPic}(\mathcal{N}(nm, tm))$  generated by  $M$ . It follows from Lemma 3 and the arguments above that  $\Psi_{\mathcal{N}(nm, tm)}(F) = \Psi_{\mathcal{N}(nm, tm)}(U)V$  for some  $V \in \text{Pic}(\mathcal{N}(m, tm))$  and some  $U \in \text{TrPic}_G(\mathcal{N}(nm, tm))$  such that  $\Phi_{\mathcal{N}(nm, tm)}(U)$  lies in the subgroup generated by  $M$ . By Theorem 3 we have  $\Phi_{\mathcal{N}(nm, tm)}(U^{-1}F) \in \text{Pic}(\mathcal{N}(nm, tm))$ . Then  $\Phi_{\mathcal{N}(nm, tm)}(U^{-1}F)$  lies in the subgroup generated by  $M$  and, consequently,  $\Phi_{\mathcal{N}(nm, tm)}(F)$  lies in the subgroup generated by  $M$ .

□

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